Notes on the p-mean curvature equation in pseudohermitian geometry.

These are the handwritten notes for my short course at the Zhejiang University summer of 2014. It is intended for undergraduate students having had some exposure to differential geometry.

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Review

Surfaces in Euclidean geometry

$$\Sigma \subset \mathbb{R}^3 \quad , \quad ds^2 = dx_1^2 + dx_2^2 + dx_3^2$$

Gauss map : $G: p \longmapsto n(p)$.

$$dn\, e_i = h_{ij} e_j \qquad h_{ij} = h_{ji}$$

$\exists$ frame $\quad dn \sim \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$

$k_1 + k_2 =$ mean curvature $\quad k_1 k_2 =$ Gauss curvature

Gauss equation $\quad k_1 k_2\, e^1 \wedge e^2 = d\omega_{12} = K\, e^1 \wedge e^2$

Codazzi equation $\quad h_{ij,k} = h_{ik,j}$

Fundamental theorem of surface.

Bernstein theorem.

$$H = \frac{(1+u_y^2)\, u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)\, u_{yy}}{(1+u_x^2+u_y^2)^{3/2}}$$

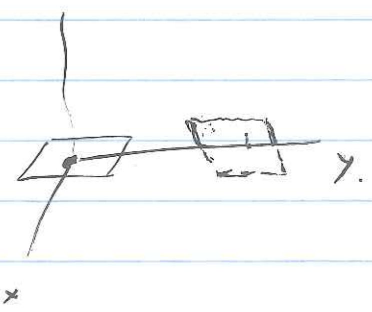
Heisenberg geometry: $\mathbb{H}^1 = \widetilde{\{(x,y,z) \in \mathbb{R}^3\}}$

$$\theta = dz + x\,dy - y\,dx$$

$$e_1 = \partial_x + y\partial_z, \quad e_2 = \partial_y - x\partial_z, \quad T = \partial_z$$

$$d\theta = 2\, dx \wedge dy. \quad \Rightarrow \quad \theta \wedge d\theta = 2\, dx \wedge dy \wedge dz$$

$$e^1 = dx, \quad e^2 = dy, \quad d\theta = 2e^1 \wedge e^2.$$



Group structure $(x\,y\,z)\cdot(x',y',z') = (x+x',\ y+y',\ z+z'+x'y-y'x)$.

Legendrian curves: $\gamma : \overset{t}{(a,b)} \to \mathbb{R}^3$.

$$\frac{dz}{dt} + x\frac{dy}{dt} - y\frac{dx}{dt} = 0$$

Contact structure: (M^3, θ) θ a nondeg 1-form
$\theta \wedge d\theta \neq 0$.

ξ : Kernel of θ. $J : \xi \to \xi$ $J^2 = -I$.
metric on ξ : $x, y \in \xi$ $\langle x, y\rangle = d\theta(x, Jy)$

1. $\exists$ unique T s.t. $\theta(T) = 1$
$0 = \mathcal{L}_T\theta = \cancel{d}\, i_T + i_T \circ d\theta \cdot = d\theta(T, \cdot)$

2. $\exists$ ~~uni~~ frame $e_1, e_2 = Je_1$ s.t.
$d\theta = 2e^1 \wedge e^2$

Tanaka-Webster : $\exists$ unique connection ∇ s.t.
1) $\nabla\langle x, y\rangle = \langle\nabla x, y\rangle + \langle x, \nabla y\rangle$
2) $\nabla T = 0$ $\nabla J = 0$ $\nabla\theta = 0$

Such a connection is called p.h. connection is def'd by connection form ω s.t.
$$\nabla e_1 = \omega \otimes e_2$$
$$\nabla e_2 = -\omega \otimes e_1$$

Torsion. $de^1 = -e^2\wedge\omega + \theta\wedge(Ae^1 + Be^2)$
$de^2 = e^1\wedge\omega + \theta\wedge(Be^1 - Ae^2)$

Recall $\mathrm{Tor}(v,w) = \nabla_v w - \nabla_w v - [v,w]$ is a tensor.
1) $\mathrm{Tor}(x,y) = -d\theta(x,y)T$
2) $\mathrm{Tor}(T, Jy) = -J\,\mathrm{Tor}(T,y)$
3) $\mathrm{Tor}(T,x) = -\frac{1}{2}J\mathcal{L}_T Jx$ $\Big\}$ $x, y \in \xi$

<u>Example</u> Heisenberg. $e_1 = \partial_x + y\partial_z$, $e_2 = \partial_y - x\partial_z$

Then $A + iB = 0$

If $\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = (e_1', e_2') = O \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ $\quad O = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

then $\omega' = d\theta + \underbrace{\cancel{\omega}}_{0} = d\theta$

<u>Curvature</u>

$$d\omega = W e^1 \wedge e^2 + (\text{Torsion term}) \wedge \theta$$

W is called Webster curvature.

<u>Complex notation</u>: $Z_1 = e_1 - ie_2$, $Z_{\bar 1} = e_1 + ie_2$

$\{\theta^1, \theta^{\bar 1}, \theta\}$ be dual to $\{Z_1, Z_{\bar 1}, T\}$ then

$$\begin{cases} d\theta = i\,\theta^1 \wedge \theta^{\bar 1} \\ d\theta^1 = \cancel{\omega_1^1 \theta}\ \theta^1 \wedge \omega_1^1 + A^1{}_{\bar 1}\, \theta \wedge \theta^{\bar 1}, \quad \omega_1^1 + \omega_{\bar 1}^{\bar 1} = 0,\ \omega_1^1 = i\omega \\ d\omega_1^1 = W\, \theta^1 \wedge \theta^{\bar 1} + 2i \operatorname{Im}(A_{11,\bar 1}\, \theta^1 \wedge \theta) \end{cases}$$

Called Struct. equations.

(1) Recall Torsion $T(v,x) = \nabla_v x - \nabla_x v - [x, v]$.
is a tensor.

Equation of geodesics:

Let $\gamma : [0, \ell] \to M^3$ be a unit speed contact curve.

Consider the energy integral

$$E(\gamma) = \int_0^\ell |\dot\gamma(s)|^2 \, ds.$$

Let v be a variation of γ : i.e. $v(t) = \frac{d}{dt}\Phi_t(\gamma(s))$
where Φ_t is a 1-p-group of diffeo associated
to v, and $v(0) = v(\ell) = 0$

$$\frac{\partial}{\partial t}\Big|_{t=0} E(\Phi_t(\gamma)) = \int_0^\ell v\langle x, x\rangle \, ds \quad \text{where } x = \frac{d\gamma}{ds}$$

$$= 2\int_0^\ell \langle \nabla_v x, x\rangle \, ds$$

$$= 2\int_0^\ell \langle \nabla_x v + Tor(v,x), x\rangle \, ds$$

$v = \theta(v)T + v'$, $v' \in \xi$

$$= 2\int_0^\ell x\langle v, x\rangle - \langle v, \nabla_x x\rangle + \langle Tor(v,x), x\rangle \, ds$$

$$= 2\int_0^\ell -\langle v, \nabla_x x\rangle + \theta(v)\langle Tor(T,x), x\rangle \, ds$$

write $\nabla_x x = \alpha Jx + y$

$$= 2\int_0^\ell -\alpha\langle v, Jx\rangle - \langle v, y\rangle + \theta(v)\langle Tor(T,x), x\rangle \, ds$$

$d\theta(v,x) = -\langle v, Jx\rangle$
$\quad\| $
$-x\,\theta(v)$

$$= 2\int_0^\ell -\alpha(x\,\theta(v)) - \langle v, y\rangle + \theta(v)\langle Tor(T,x), x\rangle \, ds$$

$$= 2\int_0^\ell \left.\begin{matrix}\langle \theta(v)[x(\alpha) + Tor(T,x)], x\rangle \\ + \langle v, y\rangle\end{matrix}\right\} ds$$

So if γ is a length minimizer, or a critical pt of the length, then

$$0 = \frac{d}{dt}\Big|_{t=1} E(\Phi_t(\gamma)) \qquad \text{for all } v$$

hence $\langle v, y\rangle = 0$ for all v. by choosing v s.t. $\theta(v)=0$

then

$$X\alpha + Tor(T, X) = 0 \quad \text{for all } v \text{ of the form } \theta(v) \neq 0.$$

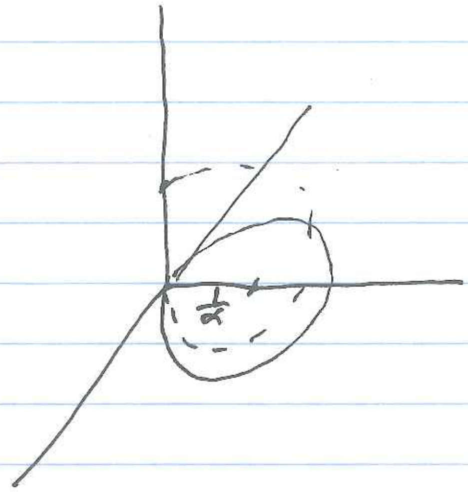
So the equation of geodesics

$$\begin{cases} \nabla_X X = \alpha J X \\ \dot{\alpha} = -\langle Tor(T,X), X\rangle \end{cases}$$

On the Heisenberg, $Tor(T, X) = 0 \quad \forall X$

$\therefore$ geodesics are contact curves for which

$$\nabla_X X = \alpha J X, \quad \alpha = \text{constant}.$$



$$dz = -x\,dy + y\,dx$$

$$\int_\gamma dz = -\text{area}(\text{⊘})$$

Pansu's sphere. Σ_α is the surface of revolution by rotating γ around z-axis

<u>Application</u>: What are the isometries of $\mathbb{H}^1$?
or maps that preserve θ.?

Let $\Phi(0) = p$, let $L_{p^{-1}}$ be left multiplication by p^{-1}.

Consider $\Psi = L_{p^{-1}} \circ \Phi$ also an isometry of $\mathbb{H}^1$ fixing 0. Then Ψ must be a rotation:

$$(x, y, z) \xrightarrow{\Psi} (\cos\alpha\, x + \sin\alpha\, y, -\sin\alpha\, x + \cos\alpha\, y, z).$$

<u>Area of a surface</u> Let $v = f e_2$ be a variation field

$$\frac{d}{dt} Vol(\Phi_t(\Omega)) = \frac{d}{dt} \int_{\Omega_t} \theta \wedge d\theta = \frac{d}{dt} \int_{\Omega} (d\Phi_t)^* \theta \wedge d\theta$$

$$= \int \mathcal{L}_v \, \theta \wedge d\theta$$

$$= \int (d \circ i_v + i_v \circ \underbrace{d) \theta \wedge d\theta}_{\overset{\parallel}{0}}$$

$$= \oint_{\Sigma} i_v (\theta \wedge d\theta) + 0$$

$$= 2 \oint_{\Sigma} i_{f e_2} \theta \wedge e^1 \wedge e^2$$

$$= 2 \int_{\Sigma} f \, \theta \wedge e^1$$

Define

$$dA = \theta \wedge e^1$$

1st Variation of area

$$\frac{d}{dt}\,\mathrm{Area}(\Phi_t(\Sigma)) = \frac{d}{dt}\int_{\Sigma_t} \theta\wedge e^1$$

$$= \int_{\Sigma} \mathcal{L}_V\, \theta\wedge e^1$$

$$= \int_{\Sigma} d\circ \underbrace{i_V\, \theta\wedge e^1}_{=0} + i_{fe_2}\left(d(\theta\wedge e^1)\right)$$

$$= \int f\, i_{e_2}\left(\cancel{2e^1\wedge e^2\wedge e^1} - \theta\wedge de^1\right).$$

$$\left(\begin{array}{l} de^1 = -e^2\wedge\omega \;+\; \theta\wedge(Ae^1+Be^2) \\ \text{Since torsion vanishes on the Heisenberg } A=B=0 \end{array}\right)$$

$$= \int f\,\theta\wedge\omega.$$

$$= \int f\,\theta\wedge\left(\omega(e_1)e^1+\omega(e_2)e^2\right)$$

$$= \int f\,\omega(e_1)\,\theta\wedge e^1 + 0$$

But $\frac{d}{dt}\,\mathrm{Area}(\Phi_t(\Sigma)) \overset{\mathrm{def}}{=} \int f H\, \theta\wedge e^1$

$$\therefore\; H = \omega(e_1).$$

Recall $\nabla e_1 = \omega \otimes e_2$

so $\nabla_{e_1} e_1 = \omega(e_1) J e_1$

So $\omega(e_1)$ = curvature of the integral curve of e_1

i.e. curvature of the contact curve along e_1.

<u>HW</u> For a graph

$$\left\{\begin{array}{l} H = \dfrac{(u_y+x)^2 u_{xx} - 2(u_y+x)(u_x-y)\,u_{xy} + (u_x-y)^2 u_{yy}}{D^3} \\[2ex] D = \sqrt{(u_x-y)^2 + (u_y+x)^2} \\[2ex] \theta \wedge e^1 = D\, dx\wedge dy. \end{array}\right.$$

$$D_x = \frac{(u_x-y)\,u_{xx} + (u_y+x)\,u_{xy}}{D}$$

Note. $\operatorname{div} N_u = H$

where N_u is the v.f. along xy plane

$$N_u = \frac{(u_x-y\,,\ u_y+x)}{D}.$$

$$\operatorname{div} N_u = \left(\frac{D^2 u_{xx} - (u_x-y)\left[(u_x-y)u_{xx} + (u_y+x)u_{xy}\right]}{D^3},\ \frac{D^2 u_{yy} - (u_y+x)\left[\qquad\right]}{\cancel{D^2(u_{xy}\ \ ) - (u_x-y)}\big[} \right)$$

Rewrite mean-curvature equation

Let $N_m = \dfrac{(u_x - y,\ u_y + x)}{D}$ $\qquad N_m^{\perp} = \dfrac{(u_y + x, -u_x + y)}{}$

$$\operatorname{div} N_m = H.$$

Thm Let $\Omega \subset \mathbb{R}^2$, $u \in \mathcal{C}^2(\Omega)$ such that

(*) $\qquad \operatorname{div} N_m = H \qquad$ in $\Omega \setminus S[u]$.

Suppose $p_0 \in S[u]$ and near p_0 $\quad |H(p)| \le \dfrac{C}{|p - p_0|}$

then either

a) p_0 is an isolated singular pt

b) $\exists\ B_\epsilon(p_0)$ s.t. $B_\epsilon(p_0) \cap S[u]$ is a $\mathcal{C}^1$ curve.

Pf Consider the Jacobian matrix of $(u_x - y, u_y + x)$ at p_0

$$U(p_0) = \begin{bmatrix} u_{xx} & u_{xy} - 1 \\ u_{xy} + 1 & u_{yy} \end{bmatrix}$$

If rank $U(p_0) = 2$, then inverse fn thm says p_0 is isolated. zero of this map.

Since rank $U(p_0) \geq 1$, The other possibility is rank $U(p_0) = 1$
There exists at most 1-direction (a_0, b_0) so that

$$F_{(a_0, b_0)}(x, y) = a_0(u_x - y) + b_0(u_y + x)$$

has $\qquad \nabla F_{(a_0, b_0)}(p_0) = 0.$

$\therefore$ For all unit vectors (a, b) transverse to (a_0, b_0)

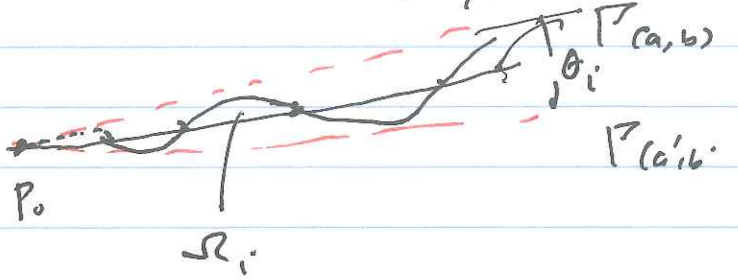
$$\Gamma_{ab} = \{ p \in \Omega \mid F_{(a,b)}(p) = 0 \}$$

is a $\mathcal{C}^1$ curve near p_0

Observe 1). $\Gamma_{(a,b)}$ has the same tangent at p_0

2) $S[u] \subset \Gamma_{(a,b)}$ near x_0

Now ~~supp~~ $S[u]$ accumulate at p_0



If $S[u]$ is not a C^1 curve near p_0 it must be like this picture

$$\left|\int_{\Omega_i} \operatorname{div} N_u \, dx\,dy\right| = \left|\int_{\Omega_i} H \, dx\,dy\right| = \left|\int_{\Omega_i} H \, r\,dr\,d\theta\right|$$

$$\leq c \int_{\Omega_i} dr\,d\theta$$

$$\leq c\,(\bar{r}_i - \underline{r}_i)\,\theta_i$$

$$= o\,(\bar{r}_i - \underline{r}_i).$$

On the other hand, LHS

$$\left|\int_{\Omega_i} \operatorname{div} N_u \, dx\,dy\right| = \left|\oint_{\partial\Omega_i} N_u \cdot \nu \, ds\right|$$

On $\Gamma_{(a,b)}$ $N_u = (-b, a)$ and $\nu = \nu_0 + o(1)$

$\Gamma_{(a',b')}$ $N_u = (-b', a')$

$$\therefore \text{LHS} \geq \left|\big((-b,a) - (-b',a')\big)\cdot \nu_0 + o(1)\right| (\bar{r}_i - \underline{r}_i)$$

$$\geq c\,(\bar{r}_i - \underline{r}_i) \qquad *$$

Assume now in addition $H = o\left(\frac{1}{r}\right)$

Improve picture near isolated singular pts.

Take Taylor expansion:

$$(u_y + x)(p) = (u_{yx}+1)(p_0)\Delta_x + u_{yy}(p_0)\Delta_y + o(|p-p_0|)$$

$$(u_x - y)(p) = u_{xx}(p_0)\Delta_x + (u_{xy}-1)(p_0)\Delta_x + o(|p-p_0|)$$

Set

$$U(p_0) = \begin{pmatrix} u_{xx} & u_{yy}+1 \\ u_{xy}-1 & u_{yy} \end{pmatrix}(p_0) = \begin{pmatrix} c & a \\ d & b \end{pmatrix}$$

$$H = \frac{P}{D^3} \quad \text{where}$$

$$P = (u_y+x)^2 u_{xx} - 2(u_y+x)(u_x-y)u_{xy} + (u_x-y)^2 u_{yy}$$

$$= (bc-ad)(\Delta x \ \Delta y)\begin{pmatrix} c & a \\ d & b \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} + o(|p-p_0|^2).$$

$$D^3 = \left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}\right|^3 + o(|p-p_0|^3).$$

Choose $\Delta y = 0$

$$H \sim \frac{(bc-ad)\, c\, |\Delta x|^2}{((a^2+c^2)|\Delta x|^2)^{3/2}} = \frac{(bc-ad)\, c}{(a^2+c^2)^{3/2}|\Delta x|} \Rightarrow c = 0$$

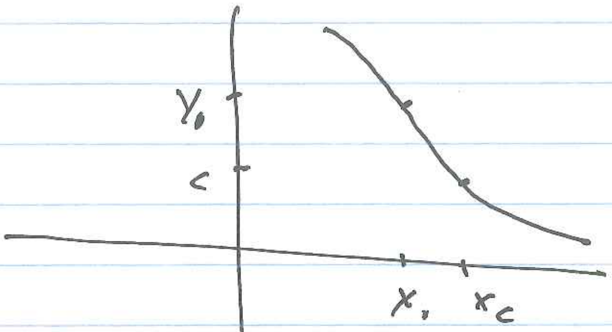
Choose $\Delta x = 0$

$$H \sim \frac{(bc-ad)\, b}{(b^2+d^2)^{3/2}\Delta y} \Rightarrow b = 0$$

So $$H \sim \frac{-ad(a+d)\Delta x \Delta y}{(a^2(\Delta x)^2 + d^2(\Delta y)^2)^{3/2}} \Rightarrow a+d = 0$$

$$\Rightarrow d(N^{-1}D)\Big|_{p_0} = \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Case Γ is a $\mathcal{C}^1$-singular curve.
Say represented as graph over y axis



Given c near y_0 $\exists\, x_c$ s.t. $(c, x_c) \in \Gamma$

$$\frac{(u_y+x)(x,c)}{(u_x-y)(x,c)} = \frac{(u_y+x)(x,c)-(u_y+x)(x_c,c)}{(u_x-y)(x,c)-(u_x-y)(x_c,c)}$$

$$= \frac{(x-x_c)(u_{xy}+1)(x_c',c)}{(x-x_c)(u_{xx})(x_c'',c)}$$

Let $(x,c) \to (x_0,y_0)$

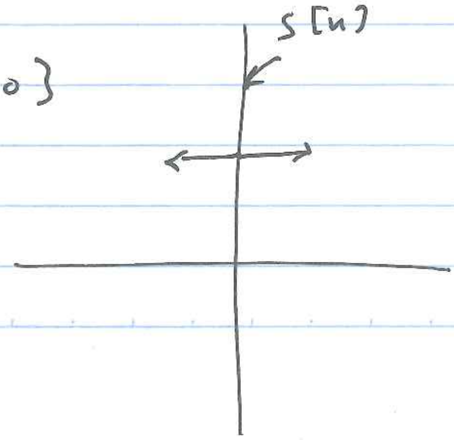
$x_c' \to x_0$ $\quad x_c'' \to x_0$

$\therefore$ The RH limit of N_u exist $=$ LH limit up to sign.

Ex $\quad u = xy$

$S[u] = \{x=0\}$

$(u_y+x, -u_x+y)$
$= (2x, 0)$



Theorem of Bernstein -

Let u be a $\mathcal{C}^2$-entire solution of the equation

$$\operatorname{div} N_u = 0 \quad \text{on } \mathbb{R}^2$$

then. u is either
1) linear function
2) $u = xy + g(y)$ after a change of coordinates.

Remark 1) $g(y)$ need only be a $\mathcal{C}^2$ function

2). Under dilation $\overset{(x', y', z')}{\delta_s}(x, y, z) = (sx, sy, s^2 z)$

$dz' + x'dy' - y'dx' = s^2\theta$.
Hence minimal surface remain minimal surface.
linear function
Of the entire solutions only
$z = 0$
and
$z = xy$
remain invariant under dilation.

$z = ax + by$
$z' = s^2 z = s^2(ax+by) = s.(ax'+by')$
So as $s \to \infty$. it converges to a vertical plane.

The Pansu sphere:

γ_λ be the geodesic:

$$x(s) = \frac{1}{2\lambda}\sin(2\lambda s) \qquad 2\lambda s = 2\pi$$
$$y(s) = \frac{1}{2\lambda}(-1 + \cos(2\lambda s)) \qquad \lambda s = \pi$$
$$z(s) = \frac{1}{2\lambda}\left(s - \frac{1}{2\lambda}\sin(2\lambda s)\right) \qquad s = \pi/\lambda$$

Rotate γ_λ around z axis

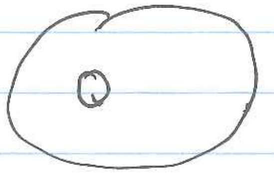
$$f(z) = \frac{1}{2\lambda^2}\left(\lambda|z|\sqrt{1-\lambda^2|z|^2} + \cos^{-1}\lambda|z|\right), \quad |z| \le \frac{1}{\lambda}$$

graph of $f(z)$ & $-f(z)$ fit to form $\mathbb{S}_\lambda$

Can check $f \in \mathcal{C}^2$ at $|z|=0$ but no more.

Suppose the minimizer of Pansu's problem is $\mathcal{C}^2$

1). $\partial\Omega$ has only one component:
If not, replace Ω by $\tilde{\Omega}$ (~~delete~~ include one of its interior) then the resulting $\tilde{\Omega}$ ~~at~~ has larger volume but smaller bdry area.



So after rescaling (to its original vol)
has a smaller bdry area.

2) $\chi(\partial\Omega) = 0$ or 2.
If $\chi = 2$, there must be an isolated singular pt p_0
Move Ω by ~~to~~ left group translation so that
0 becomes the singular pt, then $\partial\Omega = \mathbb{S}_\lambda$.

Case $\partial\Omega = T^2$ is more complicated.

Ritore: If there is a singular curve Γ, then Γ itself is a geodesic.

*.

Best result Capogna Citti - Manfredini

If Σ is the Lipschitz limit of minimizing solutions of the elliptic regularized solut.

$$\operatorname{div} \frac{(u_x - y, u_y + x)}{\sqrt{\epsilon^2 + (u_x - y)^2 + (u_y + x)^2}} = H$$

and Σ has no singular pts then Σ is $C^{1,\alpha}$

Euler equation to Pansu's prob.

$Vol(\Omega) = V$

minimize $Area(\partial\Omega)$.

Constrain $0 = \delta_{fe_2}(Vol) = \int_{\partial\Omega} f\,\theta \wedge e'$

Criticality $0 = \int f H \theta \wedge e'$

$\Rightarrow H = cst.$ at regular pts.

Elliptic approximat:

$g_L = L\theta^2 + d\theta(\cdot, J\cdot)$

Existence $\quad \operatorname{div} \dfrac{(u_x - y, u_y + x)}{\sqrt{\epsilon^2 + D^2}} = 0.$

Uniqueness:

$\Sigma_u \quad : u = x^2 + xy \qquad S[u] = y\text{-axis}$

$\Sigma_v \quad : v = xy + 1 - y^2 \qquad S[v] = \{x = y\}.$

$\Sigma_u \cap \Sigma_v$ over the unit circle

Thm $\quad$ Suppose $\Omega \subset \mathbb{R}^2$, $u, v \in \mathcal{C}^2(\bar{\Omega})$

$$\begin{cases} \operatorname{div} N(u) \geq \operatorname{div} N(v) & \text{in } \Omega \setminus S[u] \cap S[v] \\ u \leq v & \text{on } \partial\Omega \end{cases}$$

and $\quad H^1\text{-meas}\,(S[u] \cup S[v]) = 0$

then $\quad u \leq v$ on Ω

Proof $\quad$ First assume there is no singular set.

Let $\Omega_+ = \{u > v\}$.

$\boxed{\text{Claim: } N(u) = N(v) \text{ on } \Omega_+.}$

Then it follows that Ω_+ is empty:

If not, $\exists\, \epsilon > 0$ s.t.

$\Omega_{+,\epsilon} = \{u - v > \epsilon\}$ has smooth bdry Γ_ϵ

Along Γ_ϵ $\quad N^\perp(u)$ is tangent to Γ_ϵ;

$$N^{\perp}_{u} u = N^{\perp} u \cdot \nabla u = \frac{1}{D}\{(\nabla u)^{\perp} + (x,y)\} \cdot \nabla u$$

$$= \frac{1}{D}(x,y) \cdot \nabla u$$

$$= \frac{1}{D}(x,y) \cdot \{\nabla u + (-y,x)\}$$

$$= (x,y) \cdot \cancel{\nabla u}\, N u$$

Similarly

$$N^{\perp}_{v} v = (x,y) \cdot \cancel{N(v)}\, \cancel{\nabla} N_u \implies N^{\perp}_{u}(u-v) = 0.$$

But $\quad 0 = \oint_{\partial \Gamma_\epsilon} \theta = \int_{\Omega_{+,\epsilon}} d\theta > 0 \qquad *.$

Pf of claim $\quad Nu = Nv$ on Ω_+

Again try to show $Nu = Nv$ on $\Omega_{+,\delta} = \{u - v > \delta\}$.

$$I^{\delta} = \oint_{\partial\Omega_{+,\delta}} \tan^{-1}(u-v)\,(Nu - Nv) \cdot \nu^{-}\, ds$$

$$= \int_{\Omega_{+,\delta}} \left\{ \frac{1}{1+(u-v)^2} (\nabla u - \nabla v) \cdot (Nu - Nv) + \tan^{-1}(u-v)\, \mathrm{div}(Nu - Nv) \right\} dx\,dy.$$

$$\geq \int_{\Omega_{+,\delta}} \frac{1}{1+(u-v)^2} \frac{D_u + D_v}{2} \left| N(u) - N(v) \right|^2 dx\,dy.$$

Algebra: $\quad (\nabla u - \nabla v)(Nu - Nv) \cancel{\geq} \frac{D_u + D_v}{2} \left| Nu - Nv \right|^2$

$\geq c$ indep. of δ. small.

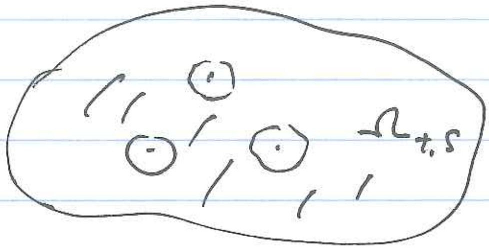
But $$I_\delta \leq \tan^{-1}\delta \underbrace{\oint (N(u)-N(v))\cdot\left(\frac{-\nabla(u-v)}{|\nabla(u-v)|}\right) ds}_{\overset{\wedge}{0}}$$

$*$

If there is singular set

Cover $S[u] \cup S[v]$ by small balls $\bigcup_i^N B_j$

When $\sum |\partial B_j| < \frac{\epsilon}{2}$.



Repeat argument to $(\Omega_{+,\delta} - \bigcup_i^N B_j)$.

$\approx$

<u>Remark</u> for the example

$u = x^2 + xy$ on ∂B_1 & $v = xy + 1 - y^2$

Pf Neither solution is a minimizer. Area $= \frac{\pi\sqrt{2}}{2}$

The minimizer is

Weak soln:

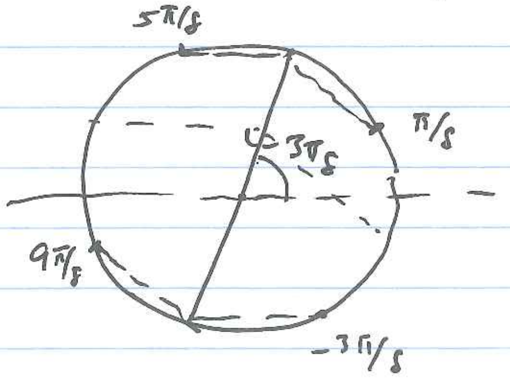
$$x = s(\sin\eta(t)) + \alpha(t)$$
$$y = -s(\cos\eta(t)) + \beta(t)$$
$$z = s[\beta(t)\sin\eta(t) + \alpha(t)\cos\eta(t)] + \gamma(t)$$

where

$$\eta(t) = \frac{\pi}{2} + \theta_2(t) - \delta(t)$$

$$\cos\left(\theta_2(t) - \frac{3}{2}\pi\right) = \frac{t}{\sqrt{2}}$$

$$\tan\delta(t) = t\sqrt{1 - t^2/2}\,/\,(1 - t^2/\sqrt{2})$$
$$\alpha(t) = t\cos 3\pi/8$$
$$\beta(t) = t\sin 3\pi/8$$
$$\gamma(t) = \cos^2 3\pi/8 + \cos 3\pi/8 \sin 3\pi/8$$



Weak Solns
1). Def.
2) Minimizer

$\Updownarrow$] Saturday

Condit for a C^2- soln to be a minimum } Monday.

Structure of C^1- soln weak. } Tuesday

Definition of weak soln $u \in W^{1,1}(\Omega)$

$$\operatorname{div} \frac{\nabla u + F}{|\nabla u + F|} = H$$

if for all $\varphi \in C_0^\infty(\Omega)$ we have

$$\int_{S[u]} |\nabla \varphi| + \int_{\Omega \setminus S[u]} N_u \cdot \nabla \varphi + \int_\Omega H\varphi \geq 0$$

where $S[u] = \{ x \in \Omega \mid \nabla u(x) + F(x) = 0$

Rem In general even when $F = (-y, x)$
there are examples where the singular set have
positive measure.

Balogh: For $C^{\frac{1}{2}*}$ hypersurface, $S[u]$ has $H^{2n+1}_b(S[u]) = 0$
But $\exists$ examples when

$$H^{2n}(S[u]) > 0$$

Recall Frobenius Theorem.

(*) $\quad (N_u - N_v)\cdot(\nabla u - \nabla v) \geq 0$

Consider the functional

$$\mathcal{F}[u] = \int_\Omega |\nabla u + F| + Hu \, dx$$

A critical pt of $\mathcal{F}$ say $u \in W^{1,1}(\Omega)$ satisfies the Euler eq

(1) $$\operatorname{div} \frac{\nabla u + F}{|\nabla u + F|} = H$$

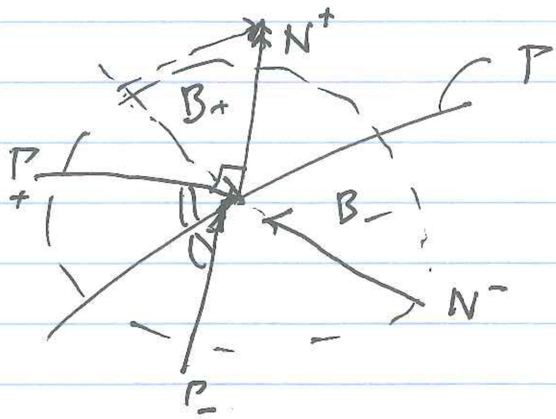
<u>Thm</u> Under the condition (*) for F, $u \in W^{1,1}(\Omega)$ is a minimizer for $\mathcal{F}[u]$ $\iff$ u is a weak soln

<u>Remark</u>

1) If $H^{n-1}(S[u]) = 0$ and u is a smooth soln to (1) then u is a minimizer

2). A C^2-solution to (1) need not be a weak soln.

In fact if $u \in C^2$ solution with Γ singular set



let ν^+ be outward normal w.r.t B_+

ν^- — — — — B_-

then $(N^+(u) - N^-(u))\cdot\nu^+$

$= (N^+(u) - N^-(u))\cdot\nu^- = 0$

Pf $\displaystyle\int_{B \setminus \Gamma} N_u \cdot \nabla\varphi + H\varphi = \left(\int_{B_+} + \int_{B_-}\right)(N_u\cdot\nabla\varphi + H\varphi)$

$$= \int_{\partial B_+} \varphi N^+(u)\cdot \nu^+ \quad \int_{\partial B_-} \varphi N^-(u), \nu^-$$

$$= \int_{P\cap B} \varphi \,(N^+u - N^-u)\cdot \nu^+$$

Now. (replace φ by $-\varphi$) $\displaystyle\int_{B\setminus P} N(u)\cdot\nabla\varphi + H\varphi = 0 \iff u$

is a weak soln

Hence $\quad (N^+u - N^-u)\cdot \nu^+ = 0. \quad$ i.e. "=" angle condition.

$\therefore \quad u = x^2 + xy. \quad$ is not a minimizer.

~

Proof of Theorem let u be a weak soln. to show that for all $\varphi \in C_0^\infty(\Omega)$, $u_\epsilon = u + \epsilon\varphi$ satisfies

$$\mathcal{F}[u_\epsilon] \geq \mathcal{F}[u].$$

Claims:

(1) $$\left.\frac{d\mathcal{F}}{d\epsilon}\right|_{\epsilon} = \pm \int_{S[u_\epsilon]} |\nabla\varphi| + \int_{\Omega\setminus S[u_\epsilon]} N_{u_\epsilon}\cdot\nabla\varphi + \int_\Omega H\varphi$$

If $\hat{n}$ is a regular value (explain in claim 3)

$\pm$ according $\begin{matrix}RH\\LH\end{matrix}$ derivative

(2) $\epsilon \mapsto \mathcal{J}[u_\epsilon]$ is a Lipschitz functio

(3) $\{\epsilon \mid \text{meas}\,(S[u_\epsilon] \cap \{|\nabla\varphi| \neq 0\} > 0)$ is a ctable set

such ϵ is called singular

(4) If $\epsilon_2 > \epsilon_1$ both regular then

then
$$\frac{d}{d\epsilon}\mathcal{J}[u_\epsilon]\Big|_{\epsilon_2} \geq \frac{d}{d\epsilon}\mathcal{J}[u_\epsilon]\Big|_{\epsilon_1}$$

(5) For any sequence of regular $\epsilon_j \searrow \hat{\epsilon}$
$$\lim_{j\to\infty} \frac{d\mathcal{J}[u_\epsilon]}{d\epsilon}\Big|_{\epsilon_j} = \frac{d\mathcal{J}}{d\epsilon}[u_\epsilon]\Big|_{\hat{\epsilon}}$$

1) 2) 3 4) 5) $\Rightarrow$ Thm.

$$\mathcal{J}[u_\epsilon] - \mathcal{J}[u] = \int_0^\epsilon \frac{d}{d\epsilon}\mathcal{J}[u_\epsilon]\,d\epsilon \geq 0.$$

Because $\frac{d}{d\epsilon}\mathcal{J}[u_\epsilon]\Big|_{\epsilon}$ (whether regular or not) $= \lim_{\substack{\epsilon_j \searrow \epsilon \\ \text{reg}}} \frac{d}{d\epsilon}\Big|_{\epsilon_j} \geq 0$

~

Pf of claim 1). & 2).

$$\mathcal{J}[u_\epsilon] = \int_{S[u_{\hat\epsilon}]} |\nabla u_\epsilon + F| + \int_{\Omega\setminus S[u_{\hat\epsilon}]} |\nabla u_\epsilon + F| + \int_\Omega H u_{\hat\epsilon} + (\epsilon-\hat\epsilon) H\varphi$$

$$= \int_{S[u_{\hat\epsilon}]} |\epsilon-\hat\epsilon||\nabla\varphi| + \int_{\Omega\setminus S[u_{\hat\epsilon}]} |\nabla u_\epsilon+F| + \int_\Omega H u_{\hat\epsilon} + (\epsilon-\hat\epsilon)H\varphi$$

$$\mathcal{J}[u_{\hat\epsilon}] = 0 + \int_{\Omega\setminus S[u_{\hat\epsilon}]} |\nabla u_{\hat\epsilon}+F| + \int_\Omega H u_{\hat\epsilon}$$

$$\frac{\mathcal{J}[u_\epsilon]-\mathcal{J}[u_{\hat\epsilon}]}{\epsilon-\epsilon'} = \frac{|\epsilon-\hat\epsilon|}{\epsilon-\hat\epsilon}\int_{S[u_{\hat\epsilon}]}|\nabla\varphi| + \int \frac{|\nabla u_\epsilon+F|-|\nabla u_{\hat\epsilon}+F|}{\epsilon-\hat\epsilon} + \int_\Omega H\varphi.$$

$$= \frac{2(\nabla u_{\hat\epsilon}+F)\cdot\nabla\varphi + (\epsilon-\hat\epsilon)|\nabla\varphi|^2}{|\nabla u_\epsilon+F| + |\nabla u_{\hat\epsilon}+F|}$$

is bdd., Hence claim 2).

Take limit $\epsilon\to\hat\epsilon$

Pf Claim (3). If $\epsilon \neq \epsilon'$ then

$S[u_\epsilon] \cap \{\nabla\varphi \neq 0\}$ and $S[u_{\epsilon'}] \cap \{\nabla\varphi \neq 0\}$ are disjoint.

Proof of claim (4).

$$\left.\frac{d\mathcal{F}[u_\epsilon]}{d\epsilon}\right|_{\epsilon_2} - \left.\frac{d\mathcal{F}[u_\epsilon]}{d\epsilon}\right|_{\epsilon_1} = \int_{\Omega \setminus (S[u_{\epsilon_1}] \cup S[u_{\epsilon_2}])} (N_{u_{\epsilon_2}} - N_{u_{\epsilon_1}}) \cdot \frac{\nabla(u_{\epsilon_2} - u_{\epsilon_1})}{\epsilon_2 - \epsilon_1} \geq 0.$$

Claim (5): Since ϵ_j is regular

$$\int_{S[u_{\epsilon_j}]} |\nabla\varphi| = 0 \Rightarrow \int_{\bigcup_1^\infty S[u_{\epsilon_j}]} N_{\epsilon_j} \cdot \nabla\varphi = 0$$

$\therefore$ Can write

$$\left.\frac{d}{d\epsilon}\mathcal{F}[u_\epsilon]\right|_{\epsilon_j} = \int_{\Omega \setminus \bigcup_1^\infty S[u_{\epsilon_k}]} N_{u_{\epsilon_j}} \cdot \nabla\varphi + \int_\Omega H\varphi$$

While in $S[u_{\hat\epsilon}] \setminus \bigcup_1^\infty S[u_{\epsilon_k}]$

$$N_{u_{\epsilon_j}} = \frac{(\nabla u_{\hat\epsilon} + F) + (\epsilon_j - \hat\epsilon_j)\nabla\varphi}{|\quad \cdot \quad|} = \frac{(\epsilon_j - \hat\epsilon)\nabla\varphi}{|\epsilon_j - \hat\epsilon||\nabla\varphi|}$$

and in $\Omega \setminus (S[u_{\hat\epsilon}] \cup \bigcup_1^\infty S[u_{\epsilon_k}])$

$$\lim_{j\to\infty} N_{u_{\epsilon_j}} = N_{u_{\hat\epsilon}}$$

Hence

$$\int_{\Omega \setminus \bigcup_{i}^{\infty} S[u_{\epsilon_i}]} N_{u_{\epsilon_j}} \cdot \nabla\varphi = \left\{ \int_{\underbrace{S[u_{\hat{\epsilon}}] \cup \bigcup_{i}^{\infty} S[u_{\epsilon_i}]}_{C}} + \int_{\Omega \setminus \underline{\quad}} \right\} N_{u_{\epsilon_j}} \cdot \nabla\varphi$$

$$\longrightarrow \int_{S[u_{\hat{\epsilon}}]} |\nabla\varphi| + \int_{\Omega \setminus S[u_{\hat{\epsilon}}]} N_{u_{\hat{\epsilon}}} \cdot \nabla\varphi$$

$\therefore$

$$\left.\frac{d\mathcal{Y}}{d\epsilon}[u_\epsilon]\right|_{\epsilon_j} = \int_{\Omega \setminus S[u_{\epsilon_j}]} N_{u_{\epsilon_j}} \cdot \nabla\varphi + \int_\Omega H\varphi$$

$$\longrightarrow \int_{S[u_{\hat{\epsilon}}]} |\nabla\varphi| + \int_{\Omega \setminus S[u_{\hat{\epsilon}}]} N_{u_{\hat{\epsilon}}} \cdot \nabla\varphi + \int H\varphi$$

$$= \left.\frac{d\mathcal{Y}}{d\epsilon}[u_\epsilon]\right|_{\hat{\epsilon}}$$

Strongly regular part of $\mathscr{C}^1$ - weak solution

$$\int_{S[u]} |\nabla\varphi| + \int_{\Omega\setminus S[u]} N_u\cdot\nabla\varphi + \int_{\Omega} H\varphi \geq 0.$$

regul.
$\Rightarrow$

$$\int_{\Omega} N_u\cdot\nabla\varphi + \int_{\Omega} H\varphi = 0.$$

Introduction of chc. coordinates:

$N = (\cos\theta, \sin\theta)$ be a $\mathscr{C}^0$ v.f satisfying

$$\int N\cdot\nabla\varphi + \int H\varphi = 0$$

for all test fns $\varphi \in \mathscr{C}_0^\infty(\Omega)$.

Goal
To introduce locally chc. coordinats. (s,t)
so that $\frac{\partial}{\partial s}$ correspond to $N^\perp$
$\frac{\partial}{\partial t}$ — — — . N

$(x,y) \longmapsto (s,t)$ is a $\mathscr{C}^1$-diffeo.

Theorem A Let Γ be a $\mathscr{C}^1$-smooth curve.

$$\begin{cases} \dfrac{dx}{d\sigma} = \sin\theta\,(x(\sigma), y(\sigma)) \\ \dfrac{dy}{d\sigma} = -\cos\theta\,(\qquad) \end{cases}$$

Then Γ is a $\mathscr{C}^2$-curve and its curvature equals $-H$.
i.e $\frac{d\theta}{d\sigma} = -H.$

Remark When N is only continuous, its integral curve need not be uniquely det'd. But in this situation it is

Theorem B: Assume H is a bounded fn on Ω. Then the integral curves to $N^\perp$ & N are unique.

Def: Chc. coordinate (s,t) satisfies

$$\nabla s \parallel N^\perp$$

$$\nabla t \parallel N$$

Assume $\operatorname{curl} F = (F_2)_x - (F_1)_y$ is smooth > 0.

Theorem C: $\exists$ function f and $g \in \mathcal{C}^0(\Omega)$ s.t.

$$\nabla s = f N^\perp, \qquad \nabla t = g D N$$

Also Nf & $N^\perp g$ exists and are continuous on $\Omega' \subset\subset \Omega$ and f, g satisfy

$$Nf + fH = 0, \qquad N^\perp g + \frac{(\operatorname{curl} F) g}{D} = 0$$

and $\Phi : (x,y) \longmapsto (s,t)$ is a $\mathcal{C}'$-diffeo

$$\Phi^* \left(\frac{ds^2}{f^2} + \frac{dt^2}{g^2 D^2} \right) = dx^2 + dy^2$$

Th Corollary: $\theta_s = \frac{\partial \theta}{\partial s} = -\frac{H}{f}$

Theorem D: Suppose NH exists & is continuous then $\theta \in \mathcal{C}'$ and the chc. curve & rec curves are $\mathcal{C}^2$ curves. In addn $N^\perp D$ exists and is continuous and

$$\theta_t = \frac{\partial \theta}{\partial t} = \frac{\operatorname{curl} F}{g D^2} - \frac{N^\perp \log D}{g D} = \frac{1}{g D^2} \left(\operatorname{curl} F - N^\perp D \right)$$

Theorem E (Codazzi equation)
θ_{st} exist & is continuous.

Corollary This implies θ_{ts} exists & ~~its~~ is continuous and
$$\theta_{st} = \theta_{ts}$$
Equating both sides, we get denote $N^{\perp}D$ by D'
$N^{\perp}N^{\perp}D$ -- D''

$$DD'' = 2\left(D' - \frac{\operatorname{curl} F}{2}\right)(D' - \operatorname{curl} F) + N^{\perp}(\operatorname{curl} F)\, D + (H^2 + NH)\, D^2$$

Remark In case of Heisenberg $\operatorname{curl} F = 2$

$$DD'' = 2(D'-1)(D'-2) + (H^2 + N(H))\, D^2$$

Proof of Thm A

Lemma: For any $\Omega' \subset\subset \Omega$ $\varphi \in \mathcal{C}^1(\Omega)$ we have
$$\oint_{\partial\Omega'} \varphi N \cdot \nu = \int_{\Omega'} \nabla\varphi \cdot N + \varphi H$$
where ν is the exterior normal to $\partial\Omega'$

Consequence: Take $\varphi \equiv 1$, $\oint_{\partial\Omega'} N \cdot \nu = \int_{\Omega'} H$

Pf of Lemma: Take $\psi \in \mathcal{C}_0^1(\Omega'')$ $\Omega' \subset\subset \Omega'' \subset \Omega$
Molify ψ to ψ_ϵ

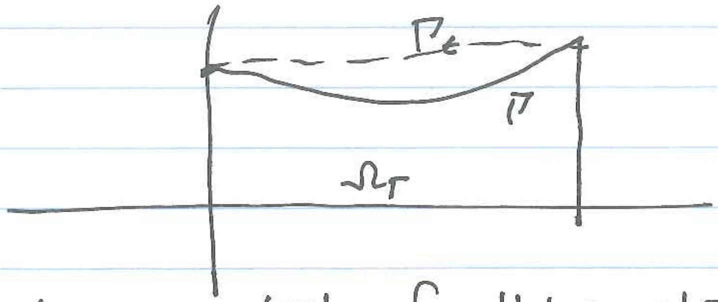
$$0 = \int_\Omega N \cdot \nabla\psi_\epsilon + H\psi_\epsilon = \int_{\Omega''} N_\epsilon \cdot \nabla\psi + H_\epsilon \psi$$

But $\operatorname{div} N_\epsilon = H_\epsilon$ in the strong sense

$$\oint_{\partial\Omega'} \varphi N_\epsilon \cdot \nu = \int (\nabla\varphi) \cdot N_\epsilon + \varphi H_\epsilon$$ Let $\epsilon \to 0$.

Pf of Thm A WLOG may assume Γ is locally a $\mathcal{C}^1$ graph

$(x, y(x))$ over $0 \le x \le a$



$\Gamma_\epsilon : (x, y(x) + \epsilon \varphi(x))$.
$\varphi \in \mathcal{C}_0^\infty([0,a])$

To show $\quad |\Gamma| - \int_{\Omega_\Gamma} H\,dx\,dy \le |\Gamma_\epsilon| - \int_{\Omega_{\Gamma_\epsilon}} H\,dx\,dy$

Let $G(\Gamma_\epsilon) = \int_{\Gamma_\epsilon} N\cdot\nu\, d\sigma - \int_{\Omega_{\Gamma_\epsilon}} H\,dx\,dy$

Then $G(\Gamma_\epsilon) - G(\Gamma)^* = \left\{ \oint_{\partial\Omega_\epsilon} N\cdot\nu\, d\sigma - \int_{\Omega_\epsilon} H\,dx\,dy \right\} - \left\{ \int_{\partial\Omega} - \int_{\Omega} \right\}$

$= 0 - 0$

$|\Gamma| - \int H\,dx\,dy = G(\Gamma) = G(\Gamma_\epsilon)$

$\uparrow$

along Γ, $\nu = N$

$\le \int_{\Gamma_\epsilon} |N\cdot\nu|\, d\sigma - \int_{\Omega_\epsilon} H\,dx\,dy$

$\le |\Gamma_\epsilon| - \int_{\Omega_{\Gamma_\epsilon}} H\,dx\,dy$

Compute : $\quad 0 \le \frac{1}{\epsilon}\left(|\Gamma_\epsilon| - |\Gamma|\right) - \frac{1}{\epsilon}\left\{ \int_{\Omega_\epsilon} H\,dx\,dy - \int_{\Omega} H\,dx\,dy \right\}$

$$\frac{1}{\epsilon}\left(|\Gamma_\epsilon| - |\Gamma|\right) = \frac{1}{\epsilon} \int_0^a \sqrt{1 + (y' + \epsilon\varphi')^2}\,dx - \sqrt{1 + y'^2}\,dx$$

$$= \int_0^a \frac{2y'\varphi' + \epsilon(\varphi')^2}{\sqrt{1 + (y' + \epsilon\varphi')^2} + \sqrt{1 + (y')^2}}\,dx \longrightarrow \int \frac{y'\varphi'\,dx}{\sqrt{1 + (y')^2}}$$

$$\frac{1}{\epsilon} \int_{\Omega_\epsilon \setminus \Omega} H\,dx\,dy = \int_0^a H(x, y(x))\, \varphi(x)\, dx$$

$$\therefore \quad \frac{d}{dx}\left(\frac{y'(x)}{\sqrt{1+(y')^2}}\right) = -H$$

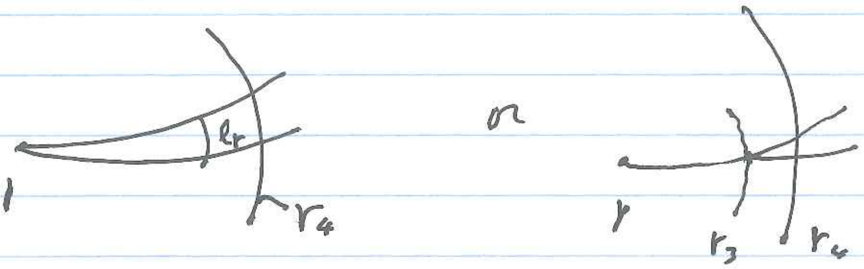
$$\Rightarrow y \in C^2$$

$\square$

<u>Theorem B</u> c) Uniqueness of ~~char.~~ seed curves

Choose $0 < r_2$ s.t. $|N(q) - N(p)| < \frac{1}{2}$ for $q \in B_{r_2}(p)$

Suppose nonuniqueness happens:



May assume $\quad N(p) \cdot \partial_r(q) \geq \frac{3}{4}$

$$\frac{1}{4} \leq N \cdot \partial_r \leq 1 \quad \text{on } l_r$$

let $\quad h(r) = \oint_{\partial\Omega_r} N \cdot \nu = \int_{\Omega_r} H$

$$\frac{1}{4}|l_r| \leq h(r) \leq |l(r)|$$

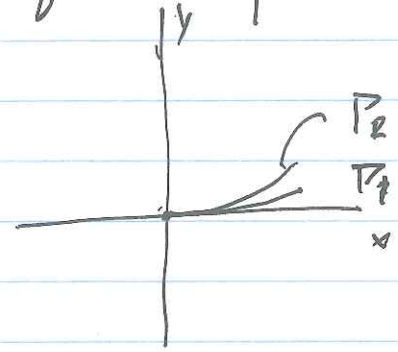
$$\therefore \quad h'(r) = \int H \leq |H|_\infty |l_r| \leq 4|H|_\infty h(r)$$

$$\therefore \quad \frac{d}{dr}\left(h(r)\, e^{-4\int_{r_3}^{r} |H|_r\, dr}\right) = [h'(r) - 4|H|_r h(r)] \cdot e^{-4\int_{r_3}^{r} |H|_r\, dr} \leq 0$$

$$\Rightarrow h(r) e^{-4\int_{r_3}^{r}} \leq h(r_3) \leq 0$$

$\ast$.

(b.) Uniqueness of the curves



$$\frac{y_j'(x)}{\sqrt{1+(y_j'(x))^2}} = -\int_0^x H(x, y_j(x))dx$$

let Γ_1 : $y_1 = y_1(x)$, $y_1(0)=0$ $y_1'(0)=0$

Γ_2 : $y_2 = y_2(x)$ $y_2(0)=0$, $y_2'(0)=0$

By continuity $\left|\frac{y_j'(x)}{1+(y_j'(x))^2}\right| \le 1/2$ for $|x| \le C$.

$$\left|\frac{y_2'(x)}{\sqrt{1+(y_2'(x))^2}} - \frac{y_1'(x)}{\sqrt{1+(y_1'(x))^2}}\right| \le \int_0^x |H(x, y_2(x)) - H(x, y_1(x))|\,dx$$
$$\le c_1 \int_0^x y_2(x) - y_1(x)\,dx$$

where c_1 is Lip. const of H.

let $h(x) \stackrel{\text{def}}{=} \int_0^x (y_2(x) - y_1(x))\,dx$

$h''(x) = y_2'(x) - y_1'(x)$

$\therefore$ $h''(x) \le c\, h(x)$.

multiply by $h'(x) > 0$ integrate

$h'(x) \le \sqrt{C_2}\, h(x)$

$h(x) \le h(\epsilon)\, e^{\sqrt{C_2}(x-\epsilon)}$

let $\epsilon \to 0$ $h(x) \equiv 0$

$\square$

First part of theorem C

$\theta \in \mathcal{C}^0(\Omega)$

$$\int_\Omega (\cos\theta, \sin\theta)\,\nabla\varphi + \int_\Omega H\varphi = 0$$

Then given $p_0 \in \Omega$ $\exists$ nbd $\Omega' \ni p_0$, $\Omega' \subset\subset \Omega$ and positive a funct $s \in \mathcal{C}^1(\Omega')$ s.t.

$$\nabla s = f N^\perp \quad \text{for some positive } f \in \mathcal{C}^0(\Omega)$$

and the level curves

$\{s = cst\}$ are reed curves

In addition

Nf exists and is continuous and

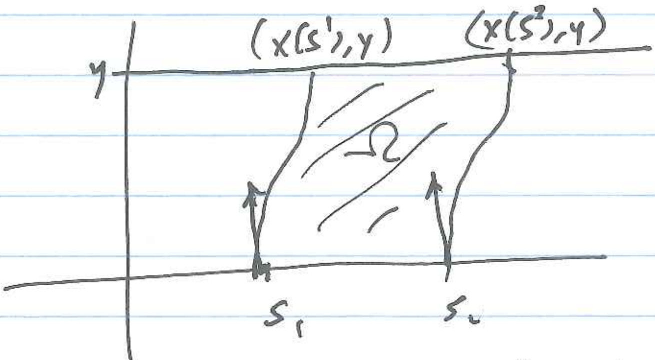
$Nf + fH = 0$

Pf WLOG assume $p_0 = (0,0)$ $N(p_0) = (0,1)$

and in $B_{r_0}(0)$, $|\theta(q) - \frac{\pi}{2}| << 1$

Let us define s on this nbd

For each $q \in B_{r_0}$, pass thru a unique reed curve



The reed curve may be parametrized by $(x(y), y)$

define $s(q) = s(p) = x$ if $p = (x, 0)$

Want to compute:

$$\frac{s(x_2,y) - s(x_1,y)}{x_2 - x_1} = \frac{s_2 - s_1}{x_2 - x_1} = \frac{s_2 - s_1}{x^{s_2}(y) - x^{s_1}(y)}$$

Consider $A(y) = \int_{x^{s_1}(y)}^{\cancel{x(s^2)}\, x^{s_2}(y)} \sin\theta(x,y)\,dx$

$$\int_{\partial\Omega} N\nu = \int_{\Omega} H.$$

Along the seed curves $\nu = N^{\perp}$

$$\text{LHS} = \int_{x^{s_1}(y)}^{x^{s_2}(y)} \sin\theta(x,y)\,dx - \int_{s_1}^{s_2} \sin\theta(x,0)\,dx$$

$$\text{RHS} = \int_0^y \left(\int_{x^{s_1}(\eta)}^{x^{s_2}(\eta)} H(s,\eta)\,ds \right) d\eta$$

$$A'(y) = \int_{x^{s_1}(y)}^{x^{s_2}(y)} H(x,y)\,dy.$$

Mean Value Thm $\Rightarrow$ $A(y) = \left(x^{s_2}(y) - x^{s_1}(y)\right) \sin\theta(\xi_1, y)$

$$A'(y) = \left(x^{s_2}(y) - x^{s_1}(y)\right) H(\xi_2, y).$$

for $x^{s_1}(y) < \xi_i < x^{s_2}(y)$

$$\frac{A'(y)}{A(y)} = \frac{H(\xi_2,y)}{\sin\theta(\xi_1,y)}$$

$$\Rightarrow \quad \frac{A(y)}{A(0)} = \exp \int_0^y \frac{H(s_2, \eta)}{\sin\theta(\eta, \eta)}\, d\eta$$

$$A(0) = (s_2 - s_1) \sin\theta(s_1, 0)$$

$$\therefore \quad \frac{s_2 - s_1}{x^{s_2}(y) - x^{s_1}(y)} = \frac{\sin\theta(s_1(y), y)}{\sin\theta(s_1(0), 0)} \exp\left\{ - \int_0^y \frac{H(s_1(\eta), \eta)}{\sin\theta(s_1(\eta)\eta)}\, d\eta \right)$$

let $x_2 \to x_1$

$$\frac{\partial s}{\partial x}(x_1, y) = \frac{\sin\theta(x, y)}{\sin\theta(s_1, 0)} \exp\left\{ - \int_0^y \frac{H(x^{s_1}(\eta), \eta)}{\sin\theta(x^{s_1}(\eta), \eta)}\, d\eta \right)$$

To compute $\frac{\partial s}{\partial y}$ observe $N\, d\tau = (dx, dy)$

where τ is length parameter along seed curve.

then

$$\frac{\partial s}{\partial y}(x_1, y) = \frac{\partial s}{\partial x}(x_1, y) \frac{\cos\theta(x_1, y)}{\sin\theta(x_1, y)}$$

$$= f(x, y) \cos\theta(x_1, y)$$

where we set

$$f(x_1, y) = \frac{1}{\sin\theta(s_1, 0)} \exp\left\{ - \int_0^y \frac{H(x^{s_1}(\eta), \eta)}{\sin\theta(x^{s_1}(\eta), \eta)}\, d\eta \right)$$

Say $\tau = 0$ at $(x_1, 0)$, $\tau = \ell$ at x_1, y then

$$f(x_1, y) = f(s_1, 0) \exp\left(- \int_0^\ell H(\gamma(t))\, dt \right.$$

where γ is the seed curve from $(s_1, 0)$

Since $\dfrac{H(x^{s_1}(y),y)}{\sin\theta(x^{s_1}(y),y)}$ is unif. bdd, f is continuous in x_1.

f is also continuous ~~in x_1~~ along γ, hence $f \in C^0(\Omega)$.

$\dfrac{\partial s}{\partial x}$ & $\dfrac{\partial s}{\partial y}$ ~~are~~ are continuous since they are given by continuous expressions.

$$\nabla s = f\cdot(\sin\theta, -\cos\theta) = fN^\perp$$

Finally $\quad Nf(x_1,y) = \dfrac{d f(\gamma_{s_1}(\tau))}{d\tau}$ at $\tau = \ell$

$$\log \frac{f(x,y)}{f(s,0)} = -\int_0^\ell H(\gamma_{s_1}(\tau))\,d\tau$$

gives $\quad Nf + fH = 0$ $\qquad \square$

2nd part of Theorem C

$\exists\, t \in C^1(\Omega')$ s.t.

$\nabla t = gDN$ for some positive $g \in C^0(\Omega')$

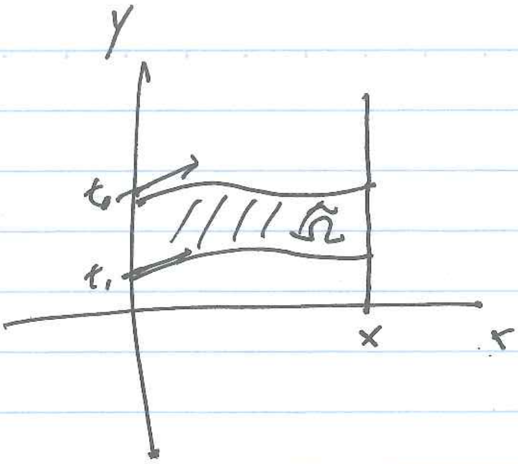
and $\{t = c\}$ are char. curves.

Moreover $N^\perp g$ exists & is continuous

$$N^\perp g + \frac{(\operatorname{curl} F)g}{D} = 0$$

Idea of proof: Similar to the 1st part.

Interchange the role of N & $N^\perp$



$$\frac{f(x,y_2)-t(x,y_1)}{y_2-y_1} = \frac{t_2-t_1}{y_2-y_1} = \frac{t_2-t_1}{y^{t_2}(x)-y^{t_1}(x)}$$

Set $B(x) = \int_{y^{t_1}(x)}^{y^{t_2}(x)} D(x,y)\sin\theta(x,y)\,dy$

Use the equat.

$$\oint_{\partial\tilde\Omega} DN^{\perp}\cdot\nu = \int_{\tilde\Omega} \operatorname{curl} F$$

to show

$$\frac{d\log B(x)}{dx} = \frac{B'(x)}{B(x)} = \frac{\operatorname{curl}\bar F(x,y)}{D(x,y)\sin\theta(x,y)}$$

□

Outline of proof of Thm D

From continuity of $(\theta_s)_t$ to show first θ_t exists.

First to show

$$\left|\underbrace{\frac{\theta(s,t_2)-\theta(s,t_1)}{t_2-t_1}}_{\text{Diff }Q}\right| \le M$$

Then observe

$$\overbrace{\left[\frac{y(s_2,t_2)-y(s_2,t_1)}{t_2-t_1} - \frac{y(s_1,t_2)-y(s_1,t_1)}{t_2-t_1}\right]}^{Q}$$

$$= \frac{y(s_2,t_2)-y(s_1,t_2)}{t_2-t_1} - \frac{y(s_2,t_1)-y(s_1,t_1)}{t_2-t_1}$$

from $\frac{dy}{ds} = \frac{-\cos\theta}{f}$

$$= \int_{s_1}^{s_2} \frac{1}{t_2-t_1}\left[\frac{\cos\theta(s,t_1)}{\cancel{t_2-t_1}\, f(s,t_1)} - \frac{\cos\theta(s,t_2)}{\cancel{t_2-t_1}\, f(s,t_2)}\right] ds$$

$$= \int_{s_1}^{s_2} (A+B)\, ds$$

Where

$$A = \frac{-1}{f(s,t_2)}\,\frac{\cos\theta(s,t_2)-\cos\theta(s,t_1)}{t_2-t_1} = \frac{(\sin\theta')}{f(s,t_2)}\,\frac{\theta(s,t_2)-\theta(s,t_1)}{t_2-t_1}$$

$$B = -\frac{\cos\theta(s,t_1)}{t_2-t_1}\left[\frac{1}{f(s,t_2)} - \frac{1}{f(s,t_1)}\right] = \frac{\cos\theta}{f^2(s,t')}\,\frac{\partial f}{\partial t}(s,t')$$

Since $A \cdot \frac{\sin\theta'}{f(s,t_2)} \ge c_1 > 0$ in $\tilde\Omega$, $|B| \le c_2$

and

$$|Q| \le 2\max|y_t| \le c_3$$

It follows that $\left|\frac{\theta(s,t_2)-\theta(s,t_1)}{t_2-t_1}\right| \le M.$

for if not, say $|(s_0)| \ge M$ then for all s close to s_0

$|(s,\cdot)| \ge M/2.$

Likewise existence of limit DiffQuot comes from writing

$$\frac{y(s_2, t_n') - y(s_2, t_1)}{t_n' - t_1} - \frac{y(s_1, t_n') - y(s_1, t_1)}{t_n' - t_1}$$

$$= \int_{s_1}^{s_2} (A_n' + B_n')\, ds$$

when $A_n' = \frac{-1}{f(s, t_n')} \frac{\cos\theta(s, t_n') - \cos\theta(s, t_1)}{t_n' - t_1}$

$$B_n' = -\frac{\cos(s, t_1)}{t_n' - t_1}\left[\frac{1}{f(s, t_n')} - \frac{1}{f(s, t_1)}\right] \to \cos\theta(s, t_1)\frac{f_t(s, t_1)}{f^2}$$

We see that

$$\lim_{n\to\infty}\int_{s_1}^{s_2} A_n'\, ds = y_t(s_2, t_1) - y_t(s_1, t_1) - \int_{s_1}^{s_2} \cos\theta(s, t_1)\frac{f_t}{f^2}(s, t_1)\,ds$$

So if $\exists\ t_n'' \to t_1$ such that

$$\frac{\theta(s_1, t_n'') - \theta(s_1, t_1)}{t_n'' - t_1} - \frac{\theta(s_2, t_n') - \theta(s_2, t_1)}{t_n' - t_1} > k$$

then this is so for s suff. close to s_1 $\left(\geq \frac{k}{2}\right)$.

But $A_n' = \frac{\sin\theta'}{f(s, t_n')} \frac{\theta(s, t_n') - \theta(s, t_1)}{t_n' - t_1}$

$$A_n'' = \frac{\sin\theta''}{\cdot\, t_n''} \frac{-\cdot\cdot\, t_n'' \cdot \cdot \cdot}{\cdot\, t_n'' \cdot \cdot \cdot}$$

Write $A_n'' - A_n' = \frac{\sin\theta''}{f(s, t_n'')}\left[\underbrace{\frac{\theta(s, t_n'') - \theta(s, t_1)}{t_n'' - t_1} - \frac{\theta(s, t_n') - \theta(s, t_1)}{t_n' - t_1}}_{\geq k}\right]$

$$+\left(\frac{\sin\theta''}{f(s, t_n'')} - \frac{\sin\theta'}{f(s, t_n')}\right)\frac{\theta(s, t_n') - \theta(s, t_1)}{t_n' - t_1}$$

$$0 \leftarrow \int_{s_1}^{s_2} A_n'' - A_n'\, ds \geq k\, C\, (s_2 - s_1) + 0 \quad *$$

Continuity of θ_t:

In the s-direction:

$$\frac{\theta(s',t')-\theta(s',t_1)}{t'-t_1}-\frac{\theta(s_1,t')-\theta(s_1,t_1)}{t'-t_1}=\int_{s_1}^{s'}\frac{\theta_s(s,t')-\theta_s(s,t_1)}{t'-t_1}\,ds$$

Take limit $t'\to t_1$ we have

(1) $$\theta_t(s',t_1)-\theta_t(s_1,t_1)=\int_{s_1}^{s'}(\theta_s)_t(s,t_1)\,ds.$$

Hence θ_t is continuous in s

In the t-direction: In (1) replace t_1 by t_m', and (1)

$$\theta_t(s',t_m')-\theta_t(s',t_1)=\theta_t(s_1,t_m')-\theta_t(s_1,t_1)$$
$$+\int_{s_1}^{s'}[(\theta_s)_t(s,t_m')-(\theta_s)_t(s,t_1)]\,ds$$

So if $\exists$ sequence $t_m'\neq t_1$ but $t_m'\to t_1$

$$\theta_t(s_1,t_m')-\theta_t(s_1,t_1)>M'>0$$

then for s' close to s_1 we have

$$\theta_t(s',t_m')-\theta_t(s',t_1)>M'/2>0$$

But we have

$$\lim_{m\to\infty}\int_{s_1}^{s'}A_m'\,ds=\psi_t(s',t_1)-\psi_t(s_1,t_1)-\int_{s_1}^{s'}\cos\theta(s,t_1)\frac{f_t}{f}(s,t_1)\,ds$$

where $$A_m'=\frac{-1}{f(s,t_m')}\,\frac{\cos\theta(s,t_m')-\cos\theta(s,t_1)}{t_m'-t_1}\xrightarrow{\text{converge}}\frac{\sin\theta(s,t_1)}{f(s,t_1)}\theta_t(s,t_1)$$

(2) $\therefore \quad y_t(s',t_1) - y_t(s_1,t_1) = \int_{s_1}^{s'} \frac{\sin\theta(s,t_1)}{f(s,t_1)} \cdot \theta_t(s,t_1)$

$$+ \int_{s_1}^{s'} \cos\theta(s,t_1) \frac{f_t}{f^2}(s,t_1)\,ds$$

Replace t_1 by t_m' in (2), take difference, and let $m \to \infty$

continuity of Y_t $$\lim_{m\to\infty} \int_{s_1}^{s'} \frac{\sin\theta(s,t_1)}{f(s,t_1)} \left[\theta_t(s,t_m') - \theta_t(s,t_1)\right] ds = 0$$

On the other hand

$$\frac{\sin\theta(s,t_1)}{f(s,t_1)} \left(\theta_t(s,t_m') - \theta_t(s,t_1)\right) \geq \frac{1}{2\max|f|} \cdot \frac{M'}{2} \quad *.$$

$\sim$

Summary: $\theta \in \mathcal{C}^1$ in (s,t) coord, hence $\mathcal{C}^1$ in (x,y).
$\therefore$ chc. & seed curves are $\mathcal{C}^2$

To compute θ_t: First $D_s \left(= f^{-1} N^2(D)\right)$

Observe $$x_s = \frac{\sin\theta}{f},\quad x_t = \frac{\cos\theta}{gD},\quad y_s = -\frac{\cos\theta}{f},\quad y_t = \frac{\sin\theta}{gD}$$

Recall calculus: $w \in \mathcal{C}^1$ of $(s,t) \in \Omega \subset \mathbb{R}^2$ s.t. $(w_s)_t$ exist and is continuous, then $(w_t)_s$ exist and $= (w_s)_t$.

So $x_{st} = x_{ts}$ and D_s exists, equating $x_{st} = x_{ts}$ & $y_{st} = y_{ts}$

$$\left[\frac{(\cos\theta)\theta_t f - f_t \sin\theta}{f^2} = -\frac{(\sin\theta)\theta_s gD - (gD)_s \cos\theta}{(gD)^2}\right] \cdot \cos\theta$$

$$\left[\frac{(\sin\theta)\theta_t f - f_t(-\cos\theta)}{f^2} = \frac{\cos\theta\,\theta_s gD - (gD)_s \sin\theta}{(gD)^2}\right] \cdot \sin\theta$$

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$$\Longrightarrow \qquad \frac{\theta_t f}{f^2} = -\frac{(gD)_s}{(gD)^2}$$

$$\theta_t = -\frac{f g_s}{g^2 D} - \frac{f D_s}{g D^2}$$

$$= \frac{(\operatorname{curl} F) g}{(gD)^2} - \frac{f D_s}{g D^2}$$

$$= \frac{\operatorname{curl} F}{g D^2} - \frac{N^\perp(\log D)}{gD}.$$

Theorem E

$$(\theta_s)_t = -\frac{1}{gD} N\left(\frac{H}{f}\right) = -\frac{1}{gD}\left(\frac{N(H)}{f} - \frac{HNf}{f^2}\right)$$

$$= -\frac{1}{fgD}\left(N(H) + H^2\right) \qquad Nf = -fH$$

is continuous, hence

$(\theta_t)_s$ exists but

$$\theta_t = \frac{1}{gD^2}(\operatorname{curl} F - D')$$

hence D'' exists and is continuous

$$(\theta_t)_s = \frac{1}{f} N^\perp(\theta_t)$$

$$= \frac{1}{fgD^2}\left(N^\perp \operatorname{curl} F - D''\right)$$

$$+ \frac{1}{f}(\operatorname{curl} F - D')\left(-\frac{N^\perp g}{g^2 D^2} - \frac{2D'}{gD^3}\right)$$

$$= \frac{1}{fgD}\left(\frac{N^\perp \operatorname{curl} F - D''}{D} + \frac{(\operatorname{curl} F - D')(\operatorname{curl} F - 2D')}{D^2}\right).$$

Equate $(\theta_s)_t = (\theta_t)_s$ we get

$$DD'' = 2\left(D' - \frac{\operatorname{curl} F}{2}\right)(D' - \operatorname{curl} F) + (N^{\perp} \operatorname{curl} F) D + (H^2 + N(H)) D^2$$

In case of Heinenberg, $H = 0$

$$D D'' = 2(D'-1)(D'-2) + \cancel{(H^2 + N(H))}$$

$$2\frac{D'}{D} = \frac{D' D''}{(D'-1)(D'-2)}$$

$$= \frac{-D''}{D'-1} + \frac{2D''}{D'-2}$$

$$= -\frac{(D'-1)'}{D'-1} + \frac{2(D'-2)'}{D'-2}$$

$$\Rightarrow \quad |D'-2|^2 = c|D'-1| D^2$$

A 2nd argument:

$$\Phi^*\Big(\underbrace{\frac{ds^2}{f^2} + \frac{dt^2}{g^2 D^2}}_{g}\Big) = dx^2 + dy^2$$

Calculate $K_g = 0 \iff$ Codazzi equat.

If $D' \neq 1$ at some non-singular pt, then $D' \neq 1$ on that entire char. line

If $D' \neq 2$ - - - - - - - - - - - - .

__Example__ a) $u = 0$, $N_u = \frac{(-y, x)}{\sqrt{x^2+y^2}}$ $N_u^{\perp} = \frac{\partial}{\partial \rho}$, $D = \sqrt{x^2+y^2}$, $D' \equiv 1$

b) $u = xy$, $N_u = \frac{(0, 2x)}{2|x|}$, $N^{\perp} = \begin{cases} (1,0) \\ (-1,0) \end{cases}$ $D = 2|x|$ $D' = 2$

In general, when $H \neq 0$

Theorem B u be C^1-weak soln on Ω. $N^\perp(\mathrm{curl}\,F)$ & $N(H)$ continuous. $p \in \Omega$ be a singular pt. Let $\Gamma : [0, \bar\rho) \to \Omega$ be a chc. curve unit speed then

a) Either $\lim_{\rho \to 0} D'(\Gamma(\rho)) = \frac{\mathrm{curl}\,F}{2}(p)$ or $(\mathrm{curl}\,F)(p)$

b) $N^\perp = \partial_\rho$

c) $p \neq q$ be singular pts in Ω, then no chc. curve Γ can join p to q.

Theorem C let p be a singular pt in Ω

Either p is an isolated singular pt

a) or there exist at least one singular curve $\gamma : [0,1] \to \Omega$ $\gamma(0) = p$.

b). $\exists$ nbd U of p s.t. for any singular pt $q \in U$ $\exists$ C^0 singular curve join q to p.

Theorem D At an isolated singular point $p \in \Omega$ there is a nbd U of p, $q \in U$ is nonsingular, Γ_q the chc. curve thru q $\underbrace{\ne}_{}$ p has to meet p

Moreover $N^\perp$ of Γ_q has a limit at p, and the map

$$\psi : \underset{\partial B_{r_0}(p)}{\overset{q}{\cap}} \longrightarrow v(q) \quad \text{is a homeo.}$$

chc. curve satisfy

$$\frac{d^2x}{d\rho^2} = H\frac{dy}{d\rho}, \quad \frac{d^2y}{d\rho^2} = -H\frac{dx}{d\rho}$$

Corollary In the situation of Thm D, the value of u near p is completely det'd by $u(p)$, $\vec F$ and H.

Thm F $\quad H^2(S[u])=0$

Thm G $\quad \mathcal{C}^1$ smooth soln ~~u=~~ u with $H=0$

a) The set of non-degenerate singular pts consist of $\mathcal{C}^1$ smooth curve

b). Equal angle condit holds.

Fundamental theory of surfaces.

Thm H Assume V is a $\mathcal{C}^\infty$ v.f, D a positive $\mathcal{C}^\infty$ funct. on a nbd U of a pt in (ξ,η) plane.

$$DD'' = 2(D'-1)(D'-2) + \lambda^2 D^2$$

then on a smaller nbd $U' \subset U$ of P.

1) $\exists$ $\mathcal{C}^\infty$ orient preserv diffeo

$$(\xi,\eta) \longrightarrow (x,y)$$

and a smooth fn $\theta = \theta(\xi,\eta)$.

$$V(\theta) = -\lambda$$

$$V(x) = \sin\theta,\ V(y) = -\cos\theta$$

In addit $\quad N(x) = \cos\theta,\ N(y) = \sin\theta \quad$ (def's N)

Satisfies

$$N(\theta) = \frac{2 - V(D)}{D}.$$

(2) $\exists$ $\mathcal{C}^\infty$ funct $\quad z = z(\xi,\eta)$ so that the graph

$$z = u(x,y) \qquad D = \sqrt{(u_x-y)^2 + (u_y+x)^2}$$

satisfies $N = (u_x - y,\ u_y + x)/D$ &

$$\operatorname{div} N = \lambda$$

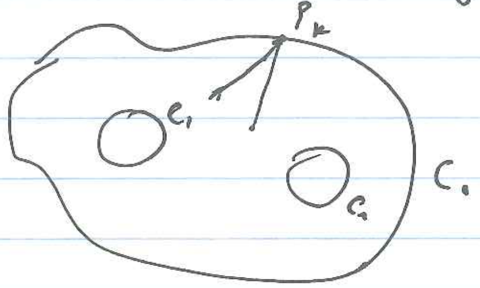
$$\operatorname{div} DV = 2$$

Where div means divergence in (x,y) coordinates.

Theorem I

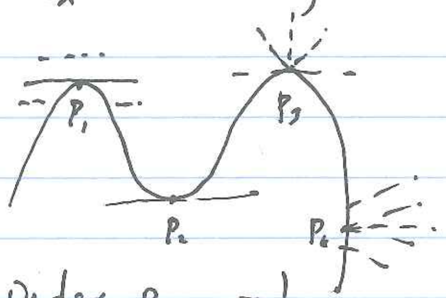
$$\chi(\Omega) = \# \pi_0(S(u)) + \sum_{j=1}^{\ell} \text{index } C_j$$

$\Omega =$



where C_j : circle

C_i has only finite many exceptional pts p_k



index $p_1 = -\frac{1}{2}$ index $p_3 = 0$

index $p_2 = \frac{1}{2}$ index $p_4 = 0$.

Proof of Theorem B $\quad \frac{m}{2} = \text{curl } F(p) \quad M = \text{curl } \bar{F}(p)$

To show given $a > 0$ $\exists \hat{\rho} > 0$ s.t.

$$D'(\Gamma(\rho)) \in (m-a, m+a) \text{ for all } 0 < \rho \le \hat{\rho}$$

or $\quad D'(\Gamma(\rho)) \in (M-a, M+a) \quad - - .$

Done by cases: $\quad$ May assume $\quad a < \frac{m}{3}$

For ρ small $\quad \rho \le \rho_0 \quad \Gamma(\rho)$ lies in a nbd V of p

$$D(\Gamma(\rho)) < \epsilon$$

$$|\text{curl } F(\Gamma_\rho) - \text{curl } \bar{F}(p)| \le \frac{a}{2}$$

ϵ taken small enough $\quad \frac{3}{4}a^2 - C_1\epsilon^2 - C_2\epsilon > 0$

~~Case I~~

$$\left.\begin{aligned} C_1 &= \sup |H^2 + N(H)| \\ C_2 &= \sup |N^\perp \text{curl } F| \end{aligned}\right\} \text{ in } V$$

Case 1 $\quad \hat{D}'(\Gamma(\rho_0)) \in (-\infty, m-a]$

If $\quad D'(\Gamma(\rho)) \in (-\infty, m-a]$ for some $\rho \le \rho_0$

then

$$D(\Gamma(\rho))D''(\Gamma(\rho)) \ge 2\frac{3}{4}a(\frac{a}{2}+1)$$

$$\ge \frac{3}{4}a^2 - C_1\epsilon^2 - C_2\epsilon \cancel{\ge 0} = C_3 > 0$$

So $\quad D'(\Gamma(\rho))$ is incr in the dirn of $N^\perp$

Say $\quad N^\perp = \frac{\partial}{\partial\rho}$ then

$$D'(\Gamma(\rho)) \in (-\infty, m-a] \text{ for } \rho \le \rho_0$$

But $\exists \rho_j \to 0$ s.t.

$$\partial_\rho D(\Gamma(\rho_j)) = D'(\Gamma(\rho_j)) > 0$$

$\therefore \quad \lim_{\rho \to 0} D'(\Gamma(\rho)) \ge 0$

We have $D'D'' \geq C_3 \frac{D'}{D}$

integrate $\left.\begin{aligned} &\tfrac{1}{2}(D')^2(\Gamma(\rho_0)) - \tfrac{1}{2}(D')^2(\Gamma(\rho)) \\ &\geq C_3\left[\log D(\Gamma(\rho_0)) - \log D(\Gamma(\rho))\right] \end{aligned}\right\}$ (1)

Let $\rho \to 0$ $\qquad \log D(\Gamma(\rho)) \to -\infty$ $\qquad$ *

If $N^\perp = -\partial_\rho$, $D'(\Gamma(\rho)) \searrow$ when $\rho \searrow 0$

Say $D'(\Gamma(\rho_1)) \leq m - a$

then

$$\lim_{\rho\to 0} D'(\Gamma(\rho)) \leq m - a$$

$$N^\perp\left(\tfrac{1}{2}(D')^2\right) = D'D'' < C_3 \frac{D'}{D} = C_3 N^\perp \log D$$

$$\therefore -\partial_\rho\left(\tfrac{1}{2}(D')^2\right) \geq C_3 \partial_\rho \log D.$$

again we get (1) $\qquad$ *.

Similarly we deduce a contradiction ~~if~~ in the cases:

$D'(\Gamma(\rho_0)) \in [m+a, M-a]$

$D'(\Gamma(\rho_0)) \in [M+a, \infty)$

We then show that in case

$D'(\Gamma(\rho_0)) \in (M-a, m+a)$

then it stays either in $(m-a, m+a)$ for $0<\rho\leq\rho_0$

or

$D'(\Gamma(\rho_5)) \in [m+a, M-a]$ for some $0<\rho_5<\rho_0$

Finally in case $D'(\Gamma(\rho_0)) \in (M-a, M+a)$ we show $D'(\Gamma(\rho)) \in (M-a, M+a)$ for all $0<\rho<\rho_0$.

This proves claim a)

Claim b) $\partial_\rho = N^\perp$ follows from a)

Claim c) Follows from b) and continuity of $N^\perp$ on Γ.

The main tool for study of local configuration of singular set is the following technical Lemma.

<u>Main Lemma</u> Let $u \in \mathcal{E}'(\Omega)$ be $\mathcal{E}'$-weak soln over convex domain Ω. If $\tilde{\Gamma}_\infty$ is a line s.t. $\Gamma_\infty = \tilde{\Gamma}_\infty \cap \Omega$ is the limit of a sequence of chc. lines Γ_j. Then Γ_∞ is a characteristic line (Indeed, Γ_∞ contains no singular point of u).

<u>Proof</u> Consider $\Gamma_\infty \setminus S[u]$ which is open in Γ_∞.
There are only 4 possibilities

1) $\Gamma_\infty \subset S[u]$.
2) $\Gamma_\infty \setminus S[u] = \Gamma^1_\infty$
3) $\Gamma_\infty \setminus S[u] = \Gamma^1_\infty \cup \Gamma^2_\infty$
4) $\Gamma_\infty \setminus S[u] = \Gamma_\infty \longrightarrow$ Nothing to prove.

In cases 1), 2), and 3) where $S[u] \cap \Gamma_\infty$ is a closed line segment have the common feature of a nonempty closed line segment which is the limit of chc curves $\{\Gamma_j\}$.

Let's parametrize this closed segment Λ by $\gamma: [-a, a] \to \Gamma$ so that $\gamma([-\epsilon, \epsilon]) = \Lambda$.
and $\gamma_j(s) \to \gamma(s)$ as $j \to \infty$
There are five possibilities for the values of $D'(\gamma_j(s))$:

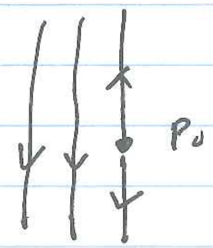
$2 < D'$, $D' = 2$, $1 < D' < 2$, $D' = 1$, $D' < 1$
Hence $\exists$ a subseq of γ_j for which D' lies in the common interval.
In each case we will arrive at a contradiction
Case 1. $D' > 2$ on Γ_j $\Rightarrow D(\gamma_j(\epsilon)) - D(\gamma_j(0)) \geq 2\epsilon$
$\Rightarrow D(\gamma(\epsilon)) - D(\gamma(0)) \geq 2\epsilon$ $\quad\divideontimes$

For the last case $D' < 1$ on Γ_j we use

$$D'' = \frac{2(D'-1)(D'-2)}{D} > 0$$

so D' is strictly increasing, then so is the case for Γ_∞.

The remaining possibility is case 3) where a singular point $p_0 \in \Gamma_\infty$ divides Γ_∞ into two open segments.



Then we have this picture, which again contradicts the continuity of $N_m^\perp$. □

The lemma can be modified for the more general situation.

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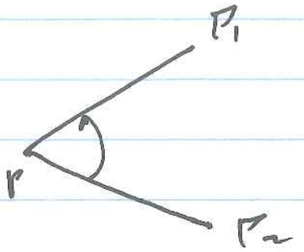
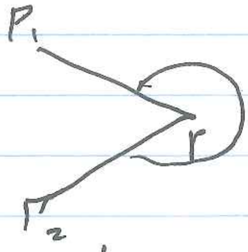
<u>Proposition</u> Let p be a singular point of Ω. Then p emits at least one char. curve.

Pf Since $S[u]$ is nowhere dense (Frobenius condition) there exists a sequence q_j of regular points converging to p. Let Γ_j be char. curves passing through q_j. each Γ_j can have only one end point, hence Γ_j limits to a char. curve issuing from p.

Corollary

Suppose $\{P_1 p P_2\}$ sector contains no singular pt

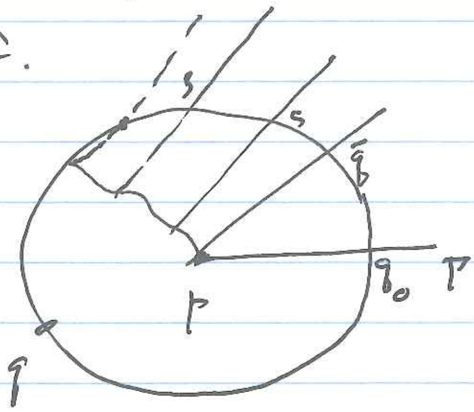




Then the sector is filled by chc. curves issuing from p.

<u>Proposition</u> Let p, q be singular pts in Ω so that $q \in \partial B_r(p)$ and $\bar{B}_r \subset \Omega$. Then $\exists$ singular curve issuing from p meeting ∂B_r.

Pf.

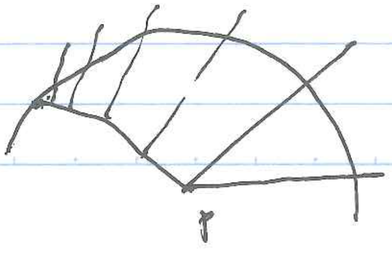


Let Γ be a chc. issuing from p, it has to reach beyond B.

Starting from q_0 go around ∂B_r. There will be a last point $\bar{q}$ so that along the arc $\overline{q_0 \bar{q}}$ each point q' passes a chc curve joining p to q'.

$\bar{q}$ is non-singular hence near $\bar{q}$ ~~too~~ also nonsingular so from a $\xi \in \partial B_r$ there passes chc curve Γ_ξ must end at a singular point other than p. Let ξ_j be a sequence of such pts $\xi_j \to \bar{\xi}$ if $\bar{\xi}$ is nonsingular then the end pts of $\Gamma_{\xi_j} \to$ end pt of $\Gamma_{\bar{\xi}}$ which must be singular. Thus we find a singular curve issuing from p.

We have either first picture or the 2nd. In both cases the singular curve reaches ∂B_r.



Proof of Theorem

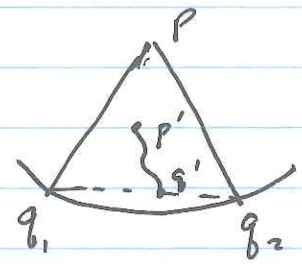
a) is given in the previous proposition.
To prove b). Let p be an ~~is~~ singular point. Take a ball $B_{r_0}(p) \subset \Omega$. Consider

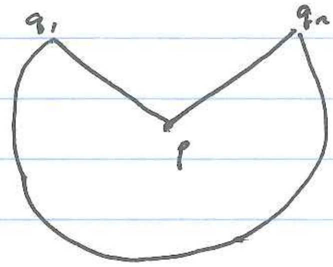
$$\Xi = \{ q \in \partial B_{r_0}(p) \mid q \notin S[u]. \text{ and the chc. line } \Gamma_q \text{ passing throug } q \text{ hits } p \}$$

We know Ξ is nonempty. To show it is closed:

Say $q_j \in \Xi$ $q_j \to q_\infty$ is regular since the Γ_{q_j} converges to Γ_∞ joining p_0 to q_∞.

$\therefore$ $\partial B_{r_0}(p) \setminus \Xi = \bigcup \{\text{open circular arcs}\}$.
Let $\widehat{q_1 q_2}$ be one such arc





q_1 is regular, hence $\exists$ nbd V of q_1 of regular pts
For $q \in V \cap \widehat{q_1 q_2}$ let $s(q)$ be the singular pt that the chc curve Γ_q ends in $\{\Gamma_1 p \Gamma_2\}$
$S(q_1) = p$, hence we found a singular curve issuing from p. in the fan region $\{\Gamma_1 p \Gamma_2\}$

If angle $\{\Gamma_1 p \Gamma_2\} < \pi$, claim any singular pt $p' \in \Delta_{q_1 p q_2}$ must connect with P.
The chc curve issuing from p' must hit $\overline{q_1 q_2}$ at q'.
Move q' along $\overline{q_1 q_2}$, the chc curve thru q' must

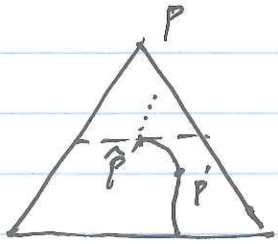
meets a singular curve γ_1 starting at p' until it hits either p or a point along $\overline{q_1 q_2}$

If the former, done

If the latter, i.e. γ_1 hits a point on $\overline{q_1 q_2}$

Consider the path compat Λ_1 containing γ_1 in $\overline{\Delta_{q_1 p q_2}}$

If $p \notin \Lambda_1$, $\exists\ \hat{p} \in \Lambda_1$ s.t. Λ_1 lies in the closed region between $q_1 q_2$ & the line $L_{\hat{p}} \parallel q_1 q_2$



Now take a seq of nonsingular pts $q_j \in \Delta_{q_1' p q_2'}$ converging to $\hat{p}$

The char. lines Γ_{q_j} converges to a char. line rising from $\hat{p}$, $\Gamma_{\hat{p}}$

$\Gamma_{\hat{p}}$ must hit either Γ_{q_1} or Γ_{q_2} or p all contradictory.

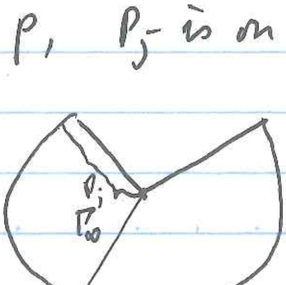
In case $\measuredangle(\Gamma_1 p \Gamma_2) > \pi$, we claim $\exists$ nbd U' of p such that any singular pt q in $U' \cap \{\Gamma_1 p \Gamma_2\}$ must connect to p via a C^0 singular curve.

Otherwise $\exists$ seq p_j of singular pts $\in \{\Gamma_1 p \Gamma_2\}$ converging to p, each of which does not connect to p.

The char. curves Γ_{p_j} issuing from p_j converge to $\Gamma_{p\infty}$ joining p.

Γ_∞ cannot join $\widehat{q_1 q_2}$, contradict $\widehat{q_1 q_2} \subset \partial B_r \setminus \Xi$

So $\Gamma_\infty = \Gamma_{q_1}$ or Γ_{q_2} but then for p_j suff close to p, p_j is on a C^0 curve issuing from p



Regularity of Singular curves

$u \in \mathcal{C}^1(\Omega)$ a $\mathcal{C}^1$-weak sol, p-minimal equation
Let U be a nbd of p s.t. $U \cap S[u]$ is a $\mathcal{C}^0$-singular curve dividing U into U_+ & U_-
We have seed curves γ_+ (γ_- resp) each pt of which issues a chc. ray in U^+ hitting a pt in a nbd of p in γ.

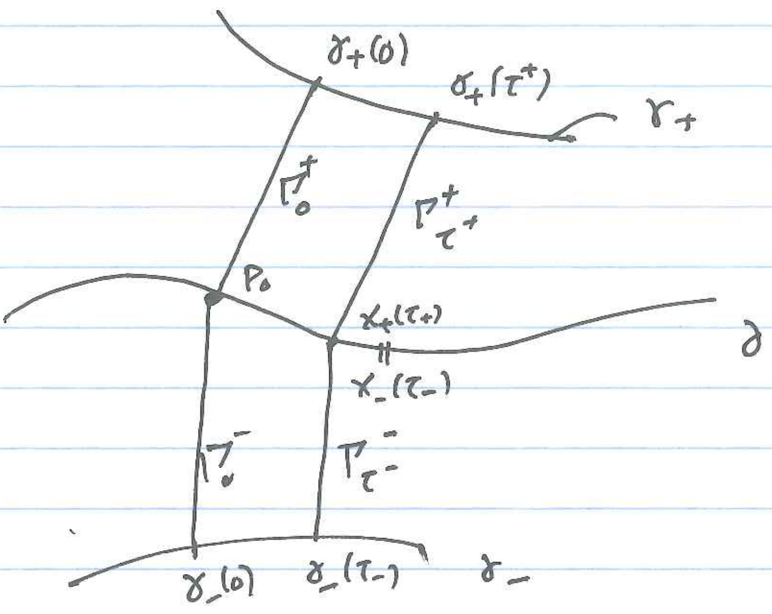
Parametrize γ_+ (γ_- resp) by the arc-length parameter τ_+ (τ_- resp) in some interval including 0

The chc ray emitted from $\gamma_+(\tau^+)$ hits γ at $X_+(\tau^+)$ ($X_-(\tau^-)$ resp.) assume $X_+(0) = X_-(0) = p_0$

Write
$$X_+(\tau^+) = \gamma_+(\tau^+) + s_+(\tau^+) N_+^\perp(\tau^+)$$
$$X_-(\tau^-) = \gamma_-(\tau^-) + s_-(\tau^-) N_-^\perp(\tau^+)$$

for some $\mathcal{C}^0$ functions $s_\pm$, where ~~$N_\pm^\perp(\tau^\pm) = (\sin\theta, -\cos\theta$~~

$$N_\pm^\perp(\tau^\pm) = (\sin\theta_\pm(\tau^\pm), -\cos\theta_\pm(\tau^\pm))$$



$$\frac{X_+(\tau^+) - X_+(0)}{\tau^+}$$

$$= \frac{\gamma_+(\tau^+) - \gamma_+(0)}{\tau^+} + \frac{s_+(\tau^+) - s(0)}{\tau^+} N_+^{\perp}(0) + s_+(\tau^+) \frac{N_+^{\perp}(\tau^+) - N_+^{\perp}(0)}{\tau^+}$$

Observe that

$$\lim_{\tau^+ \to 0} \frac{\gamma_+(\tau^+) - \gamma_+(0)}{\tau^+} = N_+(0)$$

$$\lim_{\tau^+ \to 0} \frac{N_+^{\perp}(\tau^+) - N_+^{\perp}(0)}{\tau^+} = \theta_+'(0)\, N_+(0) \qquad (\text{since } \theta_+ \in \mathcal{C}')$$

Hence

$$(1) \qquad \frac{X_+(\tau^+) - X_+(0)}{\tau^+} = \frac{s_+(\tau^+) - s_+(0)}{\tau^+} N_+^{\perp}(0) + (1 + s_+(0))\, \theta_+'(0) N_+(0) + \frac{o(\tau^+)}{\tau^+}$$

Similarly

$$(2) \qquad \frac{X_-(\tau^-) - X_-(0)}{\tau^-} = \frac{s_-(\tau^-) - s_-(0)}{\tau^-} N_-^{\perp}(0) + (1 + s_-(0)\, \theta_-'(0))\, N_-(0) + \frac{o(\tau^-)}{\tau^-}$$

Let $\chi_+(p_0) = \lim_{\tau^+ \to 0} \frac{X_+(\tau^+) - X_+(0)}{\tau^+} \cdot N_+(0) = 1 + s_+(0)\, \theta_+'(0)$

$\chi_-(p_0) = \lim_{\tau^- \to 0} \frac{X_-(\tau^-) - X_-(0)}{\tau^-} \cdot N_-(0) = 1 + s_-(0)\, \theta_-'(0)$

<u>Definition</u> We say p_0 is a non-deg. singular pt

if $\chi_+(p_0) \neq 0$ & $\chi_-(p_0) \neq 0$

<u>Remark</u> Using an integration argument one can show this def. is independent of the seed curves chosen

<u>Claim</u> If both $N_+^\perp$ & $N_-^\perp$ point toward γ (or away)

$$S_+(\gamma^+) - S_+(0) = S_-(\tau^-) - S_-(0)$$

in the opposite case $\quad S_+(\gamma^+) - S_+(0) = - S_-(\tau^-) + S_-(0)$

Apply $\quad \oint_{\partial\Omega} N_m \cdot \nu = \int_\Omega H$

where Ω is the region enclosed by the seed curves and the char. curves $\Gamma_0^+, \Gamma_0^-, \Gamma_{\tau^-}^-, \Gamma_{\tau^+}^+$

$$0 = \oint_{\partial\Omega} N \cdot \nu = \int_{\Gamma_0^+} N\cdot(-N) + \int_{\Gamma_0^-} N\cdot N + \int_{\Gamma_{\tau^-}^-} N\cdot(-N) + \int_{\Gamma_{\tau^+}^+} N\cdot N$$

$$= -S_+(0) + S_-(0) - S_-(\tau^-) + S_+(\tau^+).$$

In the limit as τ_+ & $\tau_- \to 0$ we have the existence of $\gamma'(0)$ and equal angle condition:

We claim successively

1) τ^+ is a function of τ^- for τ^- close to 0, $\lim_{\tau^- \to 0} \frac{\tau^+}{\tau^-}$ exist

2) If $\pm N_+^{\perp}(0) \neq N_-^{+}(0)$ then

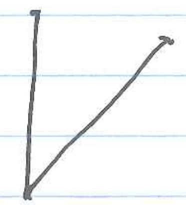
$$\lim_{\tau^- \to 0} \frac{s_-(\tau^-) - s_-(0)}{\tau^-} \quad \& \quad \lim \frac{X_-(\tau^-) - X_-(0)}{\tau^-} \text{ exists.}$$

2') If $\pm N_+^{\perp}(0) = N_-(0)$ then

$$\lim_{\tau^- \to 0} \frac{\tau^+}{\tau^-} = \pm \frac{\varkappa_-(p_0)}{\varkappa_+(p_0)}.$$

Pf

1) If $X_+(\tau_1^+) = X_+(\tau_2^+) = X(\tau)$ then we have fan region hence all Γ_+ in between ends at $X_+(\tau_1^+)$



* nondegeneracy.

multiply (1) by $\frac{\tau_+}{\tau_-}$ we have

$$(2') \qquad \frac{X_+(\tau^+) - T_+(0)}{\tau^-} = \frac{s_+(\tau^+) - s_+(0)}{\tau^-} N_+^{\perp}(0) + \frac{\tau^+}{\tau^-} \left[\varkappa_+(p_0) N_+(0) + \frac{o(\tau^+)}{\tau^+} \right]$$

LHS of (2') and (2) are the same, therefore.

$$(3) \qquad 0 = \frac{s_-(\tau^-) - s_-(0)}{\tau^-} \cdot \left(\pm N_+^{\perp}(0) - N_-^{\perp}(0) \right) + \frac{\tau^+}{\tau^-} \left[\varkappa_+(p_0) N_+(0) + \frac{o(\tau^+)}{\tau^+} \right] - \varkappa_-(p_0) N_-(0) + \frac{o(\tau^-)}{\tau^-}$$

If $N_+^{\perp}(0) \neq N_-^{\perp}(0)$, find a vector $V \perp$ to $N_+^{\pm}(0) - N_-^{\perp}(0)$

Take inner product with V

$$0 = \frac{\tau^+}{\tau^-}\left[x_+(p_0) \cdot N_+(0) \cdot V + \frac{o(\tau^+)}{\tau^+}\right]$$

$$- x_-(p_0) N_-(0) \cdot V + \frac{o(\tau^-)}{\tau^-}$$

It follows that $\lim \frac{\tau^+}{\tau^-}$ exists and

$$\lim_{\tau^- \to 0} \frac{\tau^+}{\tau^-} = \frac{x_-(p_0) N_-(0) \cdot V}{x_+(p_0) N_+(0) \cdot V}$$

Existence $\lim\limits_{\tau^- \to 0} \dfrac{s_-(\tau^-) - s_-(0)}{\tau^-}$ follows.

and then the existence of

$\lim\limits_{\tau^- \to 0} \dfrac{X_-(\tau^-) - X_-(0)}{\tau^-}$ follows from (2).

If $N_+^\perp(0) = N_-^\perp(0)$, take inner product with $N_+(0)$ in (3) to get

$$\lim_{\tau^- \to 0} \frac{\tau^+}{\tau^-} = \frac{x_-(p_0)}{x_+(p_0)}.$$

This finishes proof of Theorem G.

Local theory of surfaces of constant mean curvature in H^1

Recall $\operatorname{div} DN^\perp = \operatorname{div}(u_y + x, -(u_x - y)) = 2$

$\therefore\ DN(\theta) = 2 - N^\perp(D).$

The commutator $[N, N^\perp] = \nabla\theta$

hence
$$\nabla\theta = N(\theta)N + N^\perp(\theta)N^\perp$$
$$= \frac{2 - N^\perp(D)}{D} - \cancel{N}HN^\perp$$

Now assume everything is smooth.

Proof of Theorem H

~~(1)~~ Step 1 Solve for N transverse to V, to the equation

$$L_V N = -\frac{2 - V(D)}{D} N + \lambda V$$

Step 2 Solve for $f > 0$ and $g > 0$

$$N(f) + \lambda f = 0, \qquad V(g) + \frac{2g}{D} = 0$$

By assigning any positive initial value of f (g) along an integral curve of V (of N).

Step 3 Solve for s, and t for the equation

$(*)$ $\qquad V(s) = f, \qquad N(s) = 0$

$(**)$ $\qquad V(t) = 0 \qquad N(t) = gD.$

The integrability condition for $(*)$:

$$\left[-\left(\frac{2 - V(D)}{D}N + \lambda V\right)\right]s = L_V N(s) = VN(s) - NV(s) = 0 - N(f)$$

The integrability condition for $(**)$

$$\left[-\frac{(2 - V(D))}{D}N + \lambda V\right]t = L_V N(t) = VN(t) - NV(t) = V(gD)$$

Hence (s, t) form a coord. system.

(V, N) has the same orientation as ∂_s, ∂_t

4) Find local coord. (x, y) s.t.

$$\frac{ds^2}{f^2} + \frac{dt^2}{g^2D^2} = dx^2 + dy^2$$

So we need to check if the Gauss curvature K of the metric vanishes

From diff geometry $\quad E ds^2 + G dt^2$; $\quad A = \sqrt{EG}$

$$K = -\frac{1}{2A}\left[\partial_s \frac{\left(\frac{1}{g^2D^2}\right)_s}{\frac{1}{fgD}} + \partial_t \frac{\left(\frac{1}{f^2}\right)_t}{\frac{1}{fgD}}\right]$$

$$= -\frac{fgD}{2}\Big[\partial_s\left(-2g^{-2}D^{-1}fg_s - 2g^{-1}D^{-2}fD_s\right)$$

$$+ \partial_t\left(-2f^{-2}gDf_t\right)\Big]$$

From $(*)$ and $(**)$ we find

$$V = f\partial_s, \quad N = gD\partial_t$$

i.e. $\quad fg_s = Vg = -\frac{2g}{D}, \quad fD_s = V(D)$

$gDf_t = Nf = -\lambda f$

then we find

$$K = \frac{D''}{D} - 2D^{-2}(D'-1)(D'-2) - \lambda^2 = 0.$$

This shows the existence of (s, t).

Now it follows that V and N are orthogonal unit vectors

$$N(x) = \cos\theta \qquad N(y) = \sin\theta$$

Replace $N^\perp$ by V we get

$$[V, N] = -N(\theta)N - V(\theta)V$$

hence

$$N(\theta) = \frac{2 - V(D)}{D} \quad \text{and}$$

$$V(\theta) = -\lambda$$

Finally to find u by integrating

$$du + x\,dy - y\,dx = 0$$

By first take a ~~local~~ integral curve l of N through p $(x(p), y(p)=0)$ so that $N(u_0) + (-y, x)\cdot N = 0$.

Then integrating $du + x dy - y dx = 0$ along integral curve Γ_δ of V.

$$du + x\,dy - y\,dx \Leftrightarrow (u_x - y)\cdot dx + (u_y + x)\,dy = 0$$

*** $$\Rightarrow \quad [\nabla u + (-y, x)]\cdot V = 0$$

$$\therefore \quad \nabla u + (-y, x) = \tilde{D} N$$

for some function $\tilde{D}$

Take dot product with N:

$$\tilde{D} = N(u) + (-y, x)\cdot N = D \text{ along } l.$$

From (***) we have

$$(u_y, -u_x) + (x, y) = (\nabla u + (-y, x))^{\perp} = \tilde{D} V$$

Apply divergence to both sides, recall $V = (\sin\theta, -\cos\theta)$

$$2 = \operatorname{div}(\tilde{D} V)$$
$$= V(\tilde{D}) + \tilde{D} N(\theta)$$

$$\therefore \quad \frac{2 - V(\tilde{D})}{\tilde{D}} = N(\theta)$$

compare with $\dfrac{2 - V(D)}{D} = N(\theta)$ we find

$$V(\tilde{D}) = V(D) \text{ along } l$$

Finally differentiating

$$V\left(\frac{2-V(\tilde{D})}{\tilde{D}}\right) = V(N(\theta))$$

$$= [V,N]\theta + N\,V(\theta)$$

$$= -N(\theta)^2 - (V(\theta))^2 \qquad ([V,N] = -N(\theta)N - V(\theta)V$$

we obtain

$$\tilde{D}\tilde{D}'' = 2(\tilde{D}'-1)(\tilde{D}'-2) + 2\lambda^2\tilde{D}^2$$

We conclude $D = \tilde{D}$ etc.

we then find

$$N = \frac{\nabla u + (-y,x)}{D}$$

hence $D = |\nabla u + (-y,x)| = \sqrt{(u_x-y)^2+(u_y+x)^2}$

and $\operatorname{div} N = \lambda$

$\operatorname{div} DV = 2$ all follow.

～o～

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